Conditions for regularity and for 2-connectivity of Toeplitz graphs

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Abstract

Let $1 \le t_1 < t_2 < \cdots < t_k \le n$. A Toeplitz graph $G = (V, E)$ denoted by $T_n\langle t_1, \ldots, t_k\rangle$ is a graph where $V = \{1, \ldots, n\}$ and $E = \{(i, j) \mid |i - j| \in \{t_1, \ldots, t_k\}\}$. In this paper, we classify all regular Toeplitz graphs. Here, we present some conditions under which a Toeplitz graph has no cut-edge and cut-vertex.

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1. Introduction

A *Toeplitz matrix* is named after Otto Toeplitz (1881-1940) which is an $n \times n$ matrix $A = (a_{ij})$ such that for each i and j, $1 \le i, j \le n - 1$, $a_{ij} = a_{(i+1)(j+1)}$. Toeplitz $(0, 1)$-matrices are precisely those matrices that all diagonals parallel to main diagonal has constant values. Thus, Toeplitz matrices are determined by its first row and column. Let $n, t_1, \ldots, t_k$ be distinct positive integers such that $1 \le t_1 < t_2 < \cdots < t_k < n$. A *Toeplitz graph* is denoted by $T_n\langle t_1, \ldots, t_k\rangle = (V, E)$ where n is the number of vertices, $V = \{1, \ldots, n\}$ and $E = \{\{i, j\} \mid |i - j| \in \{t_1, \ldots, t_k\}\}$. The name of this class of graphs is due to the fact that their adjacency matrices is a

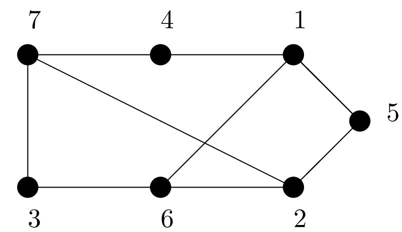

Figure 1: The Toeplitz graph $T_7 < 3, 4, 5 >$

Toeplitz $(0, 1)$-matrix. For example, see the graph $T_7 < 3, 4, 5 >$, shown in Figure 1. Moreover, the number of edges in the Toeplitz graph $T_n\langle t_1, \ldots, t_k\rangle$ is equal to $\sum_{i=1}^{k}(n - t_i)$, see [2].

Properties of Toeplitz graphs, such as biparticity, planarity, colourability and Hamiltonicity have been studied in [1]-[12].

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. We denote the degree of v in G by $d(v)$ and *minimum degree* of G is denoted by $\delta(G)$. A graph G is r-regular if $d(v) = r$, for all $v \in V(G)$. The set of neighbors of v in graph G is denoted by $N_G(v)$ or simply $N(v)$. For a set $S \subseteq V$, its open neighborhood is $N(S) = \bigcup_{v\in S} N(v)$. If G is a graph and $S \subseteq V(G)$, the induced subgraph on S is denoted by $G[S]$. The cycle of order n is denoted by C_n. A graph G is said to be *k-vertex-connected* (or *k-connected*) if the graph remains connected after deleting any fewer than k vertices from the graph. A *cut vertex* of a graph G is a vertex v whose deletion along with incident edges results in a graph with more components than the original graph. A graph is *k-edge-connected* if it remains connected whenever any fewer than k edges are removed. A *cut edge* of a graph G is an edge e whose deletion results in a graph with more components than the original graph.

2. Vertex Degrees in Toeplitz Graphs

In this section we present a result on the degree of vertices of Toeplitz graphs, and some results on the minimum degree of Toeplitz graph and characterize r-regular Toeplitz graphs.

Lemma 1 *Let $G = T_n < t >$. For even integer n and each $i \in V(G)$, $d(i) = d(n - i + 1)$. For odd integer n, $d(i) = d(n - i + 1)$ for every $i \in V(G) \setminus \{\lceil \frac{n}{2} \rceil\}$.*

Proof. Case 1. Let $t = \lfloor \frac{n}{2} \rfloor$.
For even n, we have $E(G) = \{(i,\, t+i)\,;\, 1 \leq i \leq t\}$, which clearly shows that $d(i) = 1$ for each $i \in V(G)$. Hence, for every vertex i in G, $d(i) = d(n-i+1)$. For odd n, we have $E(G) = \{(i,\, t + i)\,;\, 1 \leq i \leq t\} \cup (t + 1, n)$. Here, since $t+1$ is the only vertex with degree two, $d(i) = 1$, for every $i \in V(G)\setminus\{\lceil \frac{n}{2} \rceil\}$. Hence $d(i) = d(n - i + 1)$, for every $i \in V(G) \setminus \lceil \frac{n}{2} \rceil$.

Case 2. Let $t > \lfloor \frac{n}{2} \rfloor$.

In this case we have $E(G) = \{(i,\, t+i)\,;\, 1 \leq i \leq n-t\}$, which clearly shows that

$$d(i) = \begin{cases} 1, & for\;\; i \in \{1, 2, \ldots, n-t\} \cup \{t+1, t+2, \ldots, n\}; \\ \\ 0, & for\;\; i \in \{n-t+1, n-t+2, \ldots, t\}. \end{cases}$$

and it can be easily seen that for even n, $d(i) = d(n-i+1)$, for each $i \in V(G)$. Also for odd n, $d(i) = d(n-i+1)$, for all $i \in V(G) \setminus \{\lceil \frac{n}{2} \rceil\}$, because $d(\lceil \frac{n}{2} \rceil) = d(n - \lceil \frac{n}{2} \rceil + 1) = d(\lfloor \frac{n}{2} \rfloor + 1) = d(\lceil \frac{n}{2} \rceil)$ and there are odd number of vertices with degree zero.

Case 3. Let $t < \lfloor \frac{n}{2} \rfloor$.
In this case G consists of the paths $\{(i, i+t, i+2t, \ldots, m); 1 \leq i \leq t\}$, where m is the greatest integer less than or equal to n such that $m \cong i \pmod t$. So $d(i) \in \{1, 2\}$, for $1 \leq i \leq n$. Now, by induction on i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we will show that $d(i) = d(n-i+1)$. For $i = 1$ the result is true, because $d(1) = 1 = d(n)$. Suppose that the result holds for $i = k$, i.e., $d(k) = d(n-k+1)$. We will now show that $d(k+1) = d(n-k)$. To the contrary, suppose $d(k+1) \neq d(n-k)$. Then, without loss of generality, $d(k+1) = 2$ and $d(n-k) = 1$. Since $d(k+1) = 1$ and $t < \lfloor \frac{n}{2} \rfloor$, $(k+1)+t < n$ and $(k+1) - t < 1$. Together, these imply $k + t < n - 1$. And $k - t < 0$ and therefore $d(k) = 1$. By induction hypothesis, $d(n-k+1) = 1$. It now follows that $(n-k+1) - t = n + 1 - (k+t) > 1$ and (using the fact that $k + t < n$) $(n-k+1) + t > n$. Together, these imply $n - (k+t) > 0$ and $(n-k) + t > n - 1$. Since $d(n-k) = 2$, it must be the case that $n > k + t$ and $(n-k) + t = n$. Thus $n > 2t$, contradicting that $t < \lfloor n/2 \rfloor$, proving the Lemma." □

Proposition 1 *Let $G = T_n < t_1, \ldots, t_k >$. If n is even, then $d(i) = d(n-i+1)$ for each $i \in V(G)$. Also, if n is odd , then for each $i \in V(G) \setminus \{\lceil \frac{n}{2} \rceil\}$, $d(i) = d(n-i+1)$.*

Proof. We prove the proposition by induction on k. Lemma 1 shows that the result is true for $k = 1$. Suppose that the result is true for $k = s$, i.e., for $T_n < t_1, t_2, \ldots, t_s >$. Now we show that the result is true for $k = s+1$. We can easily see that

$$T_n < t_1, \ldots, t_{s+1} >= T_n < t_1 > \cup T_n < t_2, t_3, \ldots, t_{s+1} >$$

By induction hypothesis the result is true for both $T_n < t_1 >$ and $T_n < t_2, t_3, \ldots, t_{s+1} >$. Clearly, $E(T_n < t_1 >) \cap E(T_n < t_2, t_3, \ldots, t_{s+1} >) = \varnothing$, which implies that the result is true for $T_n < t_1 > \cup T_n < t_2, t_3, \ldots, t_{s+1} >$, which completes the proof. □

Lemma 2 *For Toeplitz graph $G = T_n < t_1, \dots, t_k >$, $\delta(G) = 0$ if and only if $t_1 \geq \lceil \frac{n+1}{2} \rceil$.*

Proof. First assume that $\delta(G) = 0$. Let $i \in V(G)$ and $d(i) = 0$. Clearly, $i + t_1 \geq n + 1$. Thus $2t_1 \geq n + 1$ and so $t_1 \geq \lceil \frac{n+1}{2} \rceil$.
Conversely, suppose that $t_1 \geq \lceil \frac{n+1}{2} \rceil$. If n is even, then $d(\frac{n}{2}) = 0$ and if n is odd then $d(\frac{n+1}{2}) = 0$. Hence $\delta(G) = 0$. □

Lemma 3 *Let $G = T_n < t_1, \dots, t_k >$ and $n \geq 2$. Then $\delta(G) = 1$ if and only if $n < t_1 + t_2$ and $t_1 < \lceil \frac{n+1}{2} \rceil$.*

Proof. First assume that $\delta(G) = 1$. Therefore, there exists $i \in V(G)$ such that $d(i) = 1$. So either $\{i, i - t_1\}$ or $\{i, i + t_1\} \in E(G)$. Without loss of generality, $\{i, i - t_1\} \in E(G)$. Since $\{i, i - t_2\}, \{i, i + t_1\} \notin E(G)$, $i \leq t_2$ and $n + 1 \leq i + t_1$. Thus $n + 1 \leq t_1 + t_2$. By Lemma 2, $t_1 < \lceil \frac{n+1}{2} \rceil$.
Conversely, suppose that $n < t_1 + t_2$ and $t_1 < \lceil \frac{n+1}{2} \rceil$. The later one implies that $\delta(G) \geq 1$. Now, we show that G has at least one vertex of degree 1. If n is even, then by assumption, we have $2t_1 \leq n$ and so $\{t_1, 2t_1\} \in E(G)$. On the other hand, clearly for integer s, $1 \leq s \leq k$, $\{t_1, t_1 - t_s\} \notin E(G)$. Since $n < t_1 + t_2$, for integer r, $2 \leq r \leq k$, $\{t_1, t_1 + t_r\} \notin E(G)$. Thus $d(t_1) = 1$. Similarly, if n is odd and $t_1 < \frac{n+1}{2}$, then $d(t_1) = 1$. □

Now, we have the following corollary.

Corollary 1 *Suppose that $G = T_n\langle t_1, \dots, t_k\rangle$ and $n \geq 2$. Then $\delta(G) \geq 2$ if and only if $n \geq t_1 + t_2$ and $t_1 < \lceil \frac{n+1}{2} \rceil$.*

Theorem 1.[7] *A Toeplitz graph $T_n < t_1, \dots, t_k >$ with $t_1 + t_k \leq n + 1$ and $gcd(t_1, \dots, t_k) = 1$ is a connected graph.*

For $k = 2$, Theorem 1 states $G = T_{t_1+t_2}\langle t_1, t_2\rangle$ is connected if $gcd(t_1, t_2) = 1$, in Proposition 2, we show the graph is d-connected if $gcd(t_1, t_2) = d$.

Proposition 2 *If $G = T_{t_1+t_2}\langle t_1, t_2\rangle$ and $gcd(t_1, t_2) = d$, then G is the union of disjoint cycles, $G_0, G_1, \dots, G_{d-1}$, and*

$$V(G_i) = \{dj - i \mid 1 \leq j \leq \frac{t_1 + t_2}{d}\},$$

$$E(G_i) = \{\{dj_1 - i, dj_2 - i\} \mid \{j_1, j_2\} \in E(T_{\frac{t_1+t_2}{d}}\langle \frac{t_1}{d}, \frac{t_2}{d}\rangle)\},$$

for $i = 0, 1, \dots, d - 1$.

Proof. Let $H = G_0 \cup G_1 \cup \cdots \cup G_{d-1}$. We claim that H is a subgraph of G. Since $d \geq 1$, for each $0 \leq i \leq d-1$ and $1 \leq j \leq \frac{t_1+t_2}{d}$, we have $1 \leq dj - i \leq t_1 + t_2$. Thus $V(H) \subseteq V(G)$. Now, we show that $E(G_i) \subseteq E(G)$. Consider the edge $\{dj_1 - i, dj_2 - i\} \in E(G_i)$. Since $|j_1 - j_2| \in \{\dfrac{t_1}{d}, \dfrac{t_2}{d}\}$ and $|dj_1 - i - (dj_2 - i)| \in \{t_1, t_2\}$, therefore $\{dj_1 - i, dj_2 - i\} \in E(G)$. Hence $E(H) \subseteq E(G)$ and we have $H \subseteq G$. Now, we prove that $E(G) \subseteq E(H)$. Suppose that $\{r, s\} \in E(G)$. Let $r \equiv -i_1 \pmod{d}$ and $s \equiv -i_2 \pmod{d}$ for $0 \leq i_1, i_2 \leq d-1$. Since $\{r, s\} \in E(G)$, $|r-s| \in \{t_1, t_2\}$. So $|r-s| \equiv 0 \pmod{d}$. Therefore, $|i_1 - i_2| \equiv 0 \pmod{d}$. Since $0 \leq i_1, i_2 \leq d-1$, $i_1 = i_2$. Assume that $i_1 = i_2 = i$. Thus, $r = dj_1 - i$ and $s = dj_2 - i$. So $j_1 = \frac{r+i}{d}$ and $j_2 = \frac{s+i}{d}$. Since $r, s \leq t_1 + t_2$, $0 \leq \frac{i}{d} < 1$ and $j_1, j_2 \leq \frac{t_1+t_2}{d}$. Hence, $\{r, s\} \in E(H)$, and we have $E(G) \subseteq E(H)$. Let $x \in V(G_a) \cap V(G_b)$ and $0 \leq a < b \leq d-1$. Therefore for some j_1 and j_2, $1 \leq j_1, j_2 \leq \frac{t_1+t_2}{d}$, $x = dj_1 - a$ and $x = dj_2 - b$. So $a \equiv b \pmod{d}$. Since $0 \leq a$, $b \leq d-1$, $a = b$. Therefore, $V(G_a) \cap V(G_b) = \varnothing$, for a and b, $0 \leq a < b \leq d-1$. Hence,

$$|V(H)| = \sum_{i=0}^{d-1} |V(G_i)| = d\,\frac{t_1+t_2}{d} = t_1 + t_2 = |V(G)|.$$

Hence $G = H$. $\square$

Now, we have the following corollary.

Corollary 2 *If $G = T_n\langle t_1, t_2\rangle$ and $n \leq t_1 + t_2$, then G is a disjoint union of paths and cycles.*

Now, we consider the Toeplitz graphs with $n \geq t_1 + t_2$ vertices.

Proposition 3 *If $G = T_n\langle t_1, t_2\rangle$, $n \geq t_1 + t_2$ and $gcd(t_1, t_2) = d$, then G is the disjoint union of $G[V_i]$, $0 \leq i \leq d-1$, where*

$$V_i = \{k(t_1+t_2) + dj - i \leq n \mid 0 \leq k \leq \lfloor \frac{n}{t_1+t_2} \rfloor \text{ and } 1 \leq j \leq \frac{t_1+t_2}{d}\}$$

for each i, $0 \leq i \leq d-1$.

Proof. Let $V_i = \{k(t_1+t_2) + dj - i \leq n \mid 0 \leq k \leq \lfloor \frac{n}{t_1+t_2} \rfloor$ and $1 \leq j \leq \frac{t_1+t_2}{d}\}$; where $d = gcd(t_1, t_2)$ and $0 \leq i \leq d-1$.
Assume that $V_a \cap V_b \neq \varnothing$, for $0 \leq a < b \leq d-1$. Then let $x \in V_a \cap V_b$. So $x = k_1(t_1+t_2) + dj_1 - a = k_2(t_1+t_2) + dj_2 - b$ for some k_1, k_2, j_1 and j_2 such that $0 \leq k_1, k_2 \leq \lfloor \frac{n}{t_1+t_2} \rfloor$, $1 \leq j_1, j_2 \leq \frac{t_1+t_2}{d}$. So $a \equiv b \pmod{d}$. Since $0 \leq a$, $b \leq d-1$, $a = b$. Hence a contradiction.
Now, suppose that $\{r, s\} \in E(G)$ and $s < r$. Let $r = k_1(t_1+t_2) + dj_1 -$

i_1 and $s = k_2(t_1+t_2) + dj_2 - i_2$, where $0 \leq i_1, i_2 \leq d-1$. So $(k_1 - k_2)(t_1+t_2) + d(j_1 - j_2) - (i_1 - i_2) \in \{t_1, t_2\}$. Since $gcd(t_1,t_2) = d$, $i_1 \equiv i_2 \pmod d$. Since $0 \leq i_1, i_2 \leq d-1$, $i_1 = i_2$ and consequently $\{r, s\} \in E(G[V_{i_1}])$. Thus $G[V_0], \ldots, G[V_{d-1}]$ are mutually disjoint subgraphs and $E(G) \subseteq \cup_{i=0}^{d-1} E(G[V_i])$.
It is straightforward to check that $V(G) = \cup_{i=0}^{d-1} V_i$. Therefore, the proof is complete. $\square$

Corollary 3 *Let $G = T_{n-1}\langle t_1, \ldots, t_k\rangle$. If $n \geq t_1 + t_k$, then $\{n-t_1, n-t_2, \ldots, n-t_k\} \subseteq V(G)$ is contained in one of the components of G.*

Proof. First, we prove that $n - t_1$ and $n - t_2$ are in the same component of $T_{n-1}\langle t_1, t_2\rangle$. Consider the notations given in the Proposition 3. Suppose that $n - t_1 \in V_{i_1}$ and $n - t_2 \in V_{i_2}$. Let $d = gcd(t_1, t_2)$. By Proposition 3, $n-t_1 = k_1(t_1+t_2)+dj_1-i_1$ and $n-t_2 = k_2(t_1+t_2)+dj_2-i_2$ for some k_1, k_2, j_1 and j_2 such that $0 \leq k_1, k_2 \leq \lfloor \frac{n-1}{t_1+t_2} \rfloor$, $1 \leq j_1, j_2 \leq \frac{t_1+t_2}{d}$ and $0 \leq i_1, i_2 \leq d$. Since $d = gcd(t_1, t_2)$, $i_1 \equiv i_2 \pmod d$. Since $0 \leq i_1, i_2 \leq d-1$, we have $i_1 = i_2$ and $n-t_1$ and $n-t_2$ are in the same component of $T_{n-1}\langle t_1, t_2\rangle$ and consequently in the same component of G. In the same way, $n-t_1$ and $n-t_i$ are in the same component of G. Therefore, the set $\{n-t_1, n-t_2, \ldots, n-t_k\}$ is contained in one of the components of G. $\square$

Theorem 2 *(i) For even k, the Toeplitz graph $G = T_n\langle t_1, \ldots, t_k\rangle$ is r-regular if and only if $r = k$, $n = t_i + t_{k-i+1}$, for each i, $1 \leq i \leq \frac{k}{2}$.*
(ii) For odd k, the Toeplitz graph $G = T_n\langle t_1, \ldots, t_k\rangle$ is r-regular if and only if $r = k$, $n = t_i + t_{k-i+1}$, for each i, $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$. Then n is even and $t_{\frac{k+1}{2}} = \frac{n}{2}$.

Proof. First, suppose that $T_n\langle t_1, \ldots, t_k\rangle$ is an r-regular graph. Obviously, $r \leq k$. Suppose that $r < k$. By the definition of Toeplitz graph, $t_{r+1} < n$. So $t_{r+1}+1 \in V(G)$ and also, $\{1, t_{r+1}+1-t_r, \ldots, t_{r+1}+1-t_1\} \subseteq N(t_{r+1}+1)$. Therefore, $d(t_{r+1}+1) \geq r+1$, a contradiction. Thus $r = k$. We claim that $n = t_1+t_k$. First, assume that $n < t_1+t_k$. Thus $d(t_1) < r$, a contradiction. Next, suppose that $n > t_1 + t_k$. Since $t_1 + t_k + 1 \leq n$, $d(t_1+1) \geq r+1$. Which is a contradiction. Hence $n = t_1 + t_k$. By induction on k, we prove the rest. The theorem is true for $k = 2$. Now, we show that the assertion holds for $k = 3$. Clearly, $T_n\langle t_1, t_2, t_3\rangle = T_n\langle t_1, t_3\rangle \cup T_n\langle t_2\rangle$. Note that $T_n\langle t_1, t_3\rangle$ and $T_n\langle t_2\rangle$ are edge disjoint. By Proposition 2, $T_n\langle t_1, t_3\rangle$ is a disjoint union of cycles. Also, $T_n\langle t_1, t_2, t_3\rangle$ is a 3-regular graph. As a result, $T_n\langle t_2\rangle$ is a perfect matching. So n is even and $t_2 = \frac{n}{2}$. Suppose that the assertion holds for $T_n\langle t_1, \ldots, t_s\rangle$, and each s, $s < k$. It is easy to see that $T_n\langle t_1, \ldots, t_k\rangle = T_n\langle t_1, t_k\rangle \cup T_n\langle t_2, \ldots, t_{k-1}\rangle$ which are edges disjoint. By Proposition 2, $T_n\langle t_1, t_k\rangle$ is a 2-regular graph. So $T_n\langle t_2, \ldots, t_{k-1}\rangle$ is a $(k-2)$-regular graph. Now, by induction hypothesis the proof of one side is complete. For the other side, first suppose that $p \in V(T_n\langle t_1, \ldots, t_k\rangle)$ and $p \leq t_1$. Since $t_1 + t_k = n$, $N(p) = \{p + t_1, \ldots, p + t_k\}$. So $d(p) = k$. Next, assume that $t_{i-1} < p \leq t_i$, for some i, $1 \leq i \leq k$. Now, we have $N(p) = \{p - t_1, \ldots, p - t_{i-1}, p + t_1, \ldots, p + t_{k-i+1}\}$ because $t_i + t_{k-i+1} = n$. Therefore, for p, $t_{i-1} < p \leq t_i$, $d(p) = k$. Finally, suppose that $t_k < p \leq n$. Clearly, $N(p) = \{p - t_1, \ldots, p - t_k\}$. Thus $T_n\langle t_1, \ldots, t_k\rangle$ is a k-regular graph. □

3. The Edge-cut and Vertex-cut in Toeplitz Graphs

In this section, we prove a necessary condition for a Toeplitz graph to be 2-edge connected.

Lemma 4 *$T_n\langle t_1, t_2\rangle$ has no cut vertex if and only if $n \geq t_1 + t_2$.*

Proof. Suppose that $n \geq t_1 + t_2$. The proof is by induction on n. If $n = t_1 + t_2$, then by Proposition 2, $T_n\langle t_1, t_2\rangle$ is the disjoint union of d cycles, where $d = gcd(t_1, t_2)$. Since cycles have no cut vertex, the assertion is true for $n = t_1 + t_2$. Clearly, the graph $T_n\langle t_1, t_2\rangle$ is constructed by adding the vertex n to $T_{n-1}\langle t_1, t_2\rangle$ and jointing n to two vertices $n - t_1$ and $n - t_2$. Since $n - 1 \geq t_1 + t_2$, the induction hypothesis shows that $T_{n-1}\langle t_1, t_2\rangle$ has no cut vertex. By Corollary 3, $n - t_1$ and $n - t_2$ are in the same component of $T_{n-1}\langle t_1, t_2\rangle$, thus $T_n\langle t_1, t_2\rangle$ has no cut vertex.

Conversely, suppose that $T_n\langle t_1, t_2\rangle$ has no cut vertex. On contrary, suppose that $n < t_1 + t_2$ but then $T_n\langle t_1, t_2\rangle$ will have a cut vertex because by Corollary 2, at least one of its components is a path which completes the proof. □

By Lemma 4 and Proposition 3, we have the following corollary.

Corollary 4 *If $gcd(t_1, t_2) = 1$ and $t_1 + t_2 \leq n$, then $T_n < t_1, t_2 >$ is 2-connected.*

Proposition 4 *Let $gcd(t_1, t_k) = 1$ and $t_1 + t_k < n$, then $T_n < t_1, \ldots, t_k >$ is 2-connected.*

Proof. Since $t_1 + t_k < n$, by Lemma 4, $T_n < t_1, t_k >$ has no cut vertex. Since $gcd(t_1, t_k) = 1$ so $gcd(t_1, \ldots, t_k) = 1$ and $t_1 + t_k < n$ which implies $t_1 + t_k \leq n + 1$, by corollary 4, $T_n < t_1, \ldots, t_k >$ is connected. $T_n < t_1, \ldots, t_k >$ has more edges than $T_n < t_1, t_k >$, so $T_n < t_1, \ldots, t_k >$ has no cut vertex, i.e., $T_n < t_1, \ldots, t_k >$ is 2-connected. □

Theorem 3 *Let n_0 be a positive. If $T_{n_0}\langle t_1, \ldots, t_k\rangle$ has no cut edge, then $T_n\langle t_1, \ldots, t_k\rangle$, $n \geq n_0$, has no cut edge as well.*

Proof. The proof is by induction on n. For $n = n_0$, there is nothing to prove. Thus assume that $n > n_0$ and $G = T_n\langle t_1, \ldots, t_k\rangle$. Suppose that the assertion holds for $n-1$, and $n > n_0$. By induction hypothesis, $T_{n-1}\langle t_1, \ldots, t_k\rangle$ as a subgraph of $T_n\langle t_1, \ldots, t_k\rangle$ has no cut edge. Therefore, the common edges of $T_{n-1}\langle t_1, \ldots, t_k\rangle$ and $T_n\langle t_1, \ldots, t_k\rangle$ are not cut edge. So it is sufficient to prove that none of the edges $\{n, n-t_i\}$, $1 \leq i \leq k$, is a cut edge. By Corollary 3, $n-t_1, n-t_2, \ldots, n-t_k$ are in the same component of $G = T_n\langle t_1, \ldots, t_k\rangle$ and we are done. □

Theorem 4 *If $3t_1 + 2t_k \leq n$, then $G = T_n < t_1, \ldots, t_k >$ has no cut edge.*

Proof. We show that $H = T_{n-t_1-t_k} < t_1, \ldots, t_k >$ has no cut edge. Assume that $1 \leq i \leq n - t_1 - t_k$. Clearly,

$$N_H(i) = \{i - t_r | 1 \leq r \leq k, t_r < i\} \cup \{i + t_r | 1 \leq r \leq k, i + t_r \leq n - t_1 + t_k\}.$$

Let r, $1 \leq r \leq k$. If $t_1 < i - t_r$, then the cycle $(i, i-t_r, i-t_r-t_1, i-t_1, i)$ is a subgraph of $T_{n-t_1-t_k} < t_1, \ldots, t_k >$ which contains the edge $\{i-t_r, i\}$. If $0 < i-t_r \leq t_1$, then $i+t_1 \leq 2t_1+t_k \leq n-t_1-t_k$, because $3t_1+2t_k \leq n$. Now, the cycle $(i, i-t_r, i-t_r+t_1, i+t_1, i)$ is a subgraph of $T_{n-t_1-t_k} < t_1, \ldots, t_k >$ which contains the edge $\{i-t_r, i\}$. If $i+t_r \leq n-t_1-t_k$ and $t_1 < i$, then the cycle $(i, i+t_r, i+t_r-t_1, i-t_1, i)$ is a subgraph of $T_{n-t_1-t_k} < t_1, \ldots, t_k >$ which contains the edge $\{i, i+t_r\}$. Since $3t_1 + 2t_k \leq n$, $i + t_r \leq 2t_1 + t_k \leq n - t_1 - t_k$. So if $i + t_r \leq n - t_1 - t_k$ and $t_1 \geq i$, then the cycle $(i, i+t_r, i+t_r+t_1, i+t_1, i)$ is a subgraph of $T_{n-t_1-t_k} < t_1, \ldots, t_k >$ which contains the edge $\{i, i+t_r\}$. Thus $T_{n-t_1-t_k} < t_1, \ldots, t_k >$ has no cut edge. Now, by Theorem 3, $G = T_n < t_1, \ldots, t_k >$ has no cut edge and the proof is complete. □

The following results were proved about the connectivity of Toeplitz graphs.

Remark 1 There is a Toeplitz graph $T_n < t_1, \ldots, t_k >$ such that $gcd(t_1, \ldots, t_k) = 1$ and $t_1 + t_k \leq n + 1$ which is connected but it is not 2-edge connected. For example $T_7 < 3, 5 >$.

Theorem 5 *If $gcd(t_1, t_k) = 1$ and $t_1 + t_k \leq n$, then $T_n < t_1, \ldots, t_k >$ is 2-edge connected.*

Proof. By Corollary 4, $T_n < t_1, \ldots, t_k >$ is a connected graph. Since $gcd(t_1, t_k) = 1$, by Proposition 2, $T_{t_1+t_k} < t_1, t_k >$ is a cycle. So $T_{t_1+t_k} < t_1, t_k >$ is 2-edge connected. Now, Theorem 3 shows that $T_n < t_1, t_k >$ has no cut edge. Thus the Toeplitz graph $T_n < t_1, \ldots, t_k >$ is 2-edge connected. □

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