Hamiltonian Cycles in Directed Toeplitz Graphs

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Abstract. An $(n \times n)$ matrix $A = (a_{ij})$ is called a Toeplitz matrix if it has constant values along all diagonals parallel to the main diagonal. A directed Toeplitz graph is a digraph with Toeplitz adjacency matrix. In this paper we discuss conditions for the existence of hamiltonian cycles in directed Toeplitz graphs.

Keywords: *Toeplitz graph; Hamiltonian graph.*

1 Introduction

A *directed* or *undirected Toeplitz graph* is, by definition, a graph with a Toeplitz adjacency matrix. A *Toeplitz matrix*, named so after Otto Toeplitz (1881-1940), is a square matrix which has constant values along all diagonals parallel to the main diagonal, i.e., a matrix of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \dots & a_{-1} & a_0 \end{pmatrix}.$$

Thus, Toeplitz matrices are defined by $2n-1$ numbers a_i, where $i = -n+1, \dots, -1, 0, 1, \dots, n-1$.

A Toeplitz matrix is called a *circulant matrix* if $a_i = a_{i-n}$ $(i = 1, \dots, n-1)$.

A directed or undirected graph whose adjacency matrix is circulant is called *circulant*.

Circulant graphs and their properties such as connectivity, hamiltonicity, bipartiteness, planarity and colourability have been studied by several authors (see [1], [2], [4], [9], [11], [12], [13] and [8]). In particular, the conjecture of Boesch and Tindell [1], that all undirected connected circulant graphs are hamiltonian, was proved by Burkard and Sandholzer [2].

Properties of Toeplitz graphs, such as bipartiteness, planarity and colourability, have been studied in [5], [6], [7]. Hamiltonian properties of undirected Toeplitz graphs have been investigated in [3] and [10]. We intend here to extend this study to the directed case. The main purpose of this paper is to offer sufficient conditions for the existence of hamiltonian cycles in directed Toeplitz graphs. Since all graphs from now on will be directed, we shall omit mentioning it. We shall consider here just graphs without multiple edges and without loops, because multiple edges and loops play no role in hamiltonicity investigations.

For a graph G, as usual, $V(G)$ will denote its vertex set and $E(G)$ its edge set. A graph C with $V(C) = \{v_1, \ldots, v_n\}$ and $E(C) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$ is called a *cycle.* (Of course, $v_i \neq v_j$ for all distinct i, j.) A cycle minus one edge is called a *path.* A graph G' is called a *subgraph* of G if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. If moreover $V(G') = V(G)$, G' is said to *span* G. If G' spans G and is a cycle or a path, it is also called *hamiltonian.* Any graph possessing a hamiltonian cycle is itself called *hamiltonian*, too.

The main diagonal of an $(n \times n)$ Toeplitz adjacency matrix will be labeled 0 and it contains only zeros. The $n-1$ distinct diagonals above the main diagonal will be labeled $1, 2, \ldots, n-1$ and those under the main diagonal will also be labeled $1, 2, \ldots, n-1$. Let $s_1, s_2, \ldots, s_k$ be the upper diagonals containing ones and $t_1, t_2, \ldots, t_l$ be the lower diagonals containing ones, such that $0 < s_1 < s_2 < \cdots < s_k < n$ and $0 < t_1 < t_2 < \cdots < t_l < n$. Then, the corresponding Toeplitz graph will be denoted by $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$. That is, $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$ is the graph with vertices 1, 2, $\ldots, n$, in which the edge (i, j) occurs if and only if $j-i = s_p$ or $i-j = t_q$ for some p and q $(1 \leq p \leq k, 1 \leq q \leq l)$. The edges of $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$ are of two types: *increasing edges* (u, v), for which $u < v$, and *decreasing edges* (u, v), where $u > v$. We define the *length* of an edge (u, v) to be $|u-v|$. Moreover, we define $[u, v] = \{u, u+1, \ldots, v-1, v\}$, where $u, v \in \mathbb{Z}^+$. Clearly, any increasing edge has length s_p for some p, and any decreasing edge has length t_q for some q.

Suppose H is a hamiltonian cycle in $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$. Then the vertices 1 and n determine in H two paths: $H_{1\to n}$ from 1 to n and $H_{n\to 1}$ from n to 1. The hamiltonian cycle H is of course determined by the paths $H_{1\to n}$ and $H_{n\to 1}$.

Remark that $T_n\langle s_1, \ldots, s_i; t_1, \ldots, t_j\rangle$ and $T_n\langle t_1, \ldots, t_j; s_1, \ldots, s_i\rangle$ are obtained from each other by reversing the orientation of all edges.

Connectivity and hamiltonicity results obtained in the undirected case for a Toeplitz graph have a direct impact on the directed case. So connectedness of $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$ means precisely connectedness of $T_n\langle s_1, \ldots, s_k,$

$t_1, \dots, t_l\rangle$ (with duplicates dropped). Hamiltonicity of $T_n\langle t_1, t_2, \dots, t_i\rangle$ means hamiltonicity of $T_n\langle t_1, \dots, t_i; t_1, \dots, t_i\rangle$.

2 Toeplitz graphs with $s_2 = 2$

Toeplitz graphs with $s_1 = 1$ (or $t_1 = 1$) obviously have a hamiltonian path. The Toeplitz graph $T_n\langle 1; n-1\rangle$ is itself a cycle for all $n \geq 1$. In this section we ask moreover that $s_2 = 2$.

We characterize here the hamiltonian Toeplitz graphs $T_n\langle 1, 2; t_1\rangle$ with even t_1 or $t_1 = 3$. For $t_1 = 5$ we establish that, with finitely many exceptions, all $T_n\langle 1, 2; 5\rangle$ are hamiltonian. Similar results are obtained for all odd $t_1 \geq 7$.

Theorem 1. *$T_n\langle 1, 2; t_1\rangle$ is hamiltonian if $n \equiv \pm 1 \,(mod\ t_1)$ and $t_1 \geq 4$.*

Proof. Suppose $n = qt_1 + 1$, where $q \in \mathbb{Z}^+$. Then a hamiltonian cycle in $T_n\langle 1, 2; t_1\rangle$ is

$$(1,\ 2, \dots, t_1,\ t_1{+}2,\ t_1{+}3, \dots, 2t_1,\ 2t_1{+}2,\ 2t_1{+}3, \dots, (q{-}1)t_1,\ (q{-}1)t_1{+}2,$$
$$(q{-}1)t_1{+}3, \dots,\ qt_1,\ qt_1{+}1,\ (q{-}1)t_1{+}1,\ (q{-}2)t_1{+}1, \dots, 2t_1{+}1,\ t_1{+}1,\ 1)$$

(see Figure 1 for the case $t_1 = 5$ and $q = 3$).

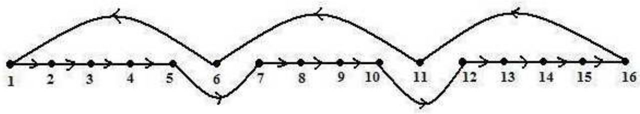

Fig. 1. A hamiltonian cycle in $T_{16}\langle 1, 2; 5\rangle$.

Next let $n = rt_1 - 1$, where $r \in \mathbb{Z}^+$. Then a hamiltonian cycle in $T_n\langle 1, 2; t_1\rangle$ is

$$(1,\ 2, \dots, t_1-2,\ t_1,\ t_1+2,\ t_1+3, \dots, 2t_1-2,\ 2t_1,\ 2t_1+1, \dots, 3t_1-2,\ 3t_1,$$
$$3t_1+1, \dots, (r-1)t_1-2,\ (r-1)t_1,\ (r-1)t_1+1, \dots,\ rt_1-1,\ (r-1)t_1-1,$$
$$\dots, 2t_1-1,\ t_1-1,\ t_1+1,\ 1)$$

(see Figure 2 for the case $t_1 = 5$ and $r = 4$). □

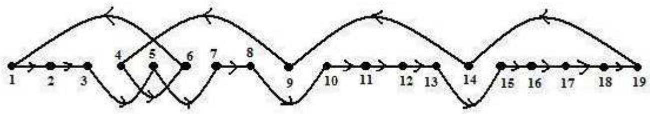

Fig. 2. $T_{19}\langle 1, 2; 5\rangle$.

Theorem 2. *For even* t_1, $T_n\langle 1, 2; t_1\rangle$ *is hamiltonian if and only if* n *is odd.*

Proof. Let $t_1 = 2m$; $m \in \mathbb{Z}^+$. Suppose n is odd and let $n \equiv n_0 \ (mod\ 2m)$ where $2m < n_0 < 4m$. Then $n_0 - 2m$ is odd.

First assume that $n = n_0$. If $n_0 - 2m = 1$, then $T_{n_0}\langle 1, 2; 2m\rangle$ has a hamiltonian cycle by Theorem 1, the proof of which shows that the cycle contains the edge $(n_0 - 1,\ n_0)$.

For $n_0 - 2m > 1$, a hamiltonian cycle in $T_{n_0}\langle 1, 2; 2m\rangle$ is

$$(1,\ 2, \ldots, n_0 - 2m - 1,\ n_0 - 2m + 1, \ldots, 2m,\ 2m + 2,\ 2m + 3, \\ \ldots,\ \underline{n_0 - 1,\ n_0},\ n_0 - 2m,\ n_0 - 2m + 2, \ldots, 2m - 1,\ 2m + 1,\ 1)$$

(see Figure 3).

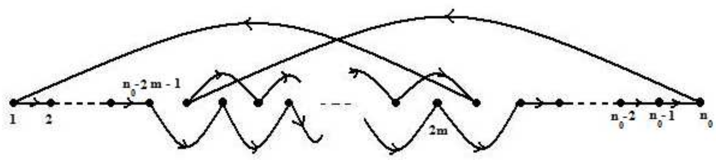

Fig. 3. $T_{n_0}\langle 1, 2; t_1\rangle$, for even t_1.

We have underlined the vertices $n_0 - 1,\ n_0$ to emphasize that $(n_0 - 1, n_0)$ is an edge in the hamiltonian cycle. For $n = n_0 + 2m$, we have $4m < n < 6m$. We transform the hamiltonian cycle in $T_{n_0}\langle 1, 2; 2m\rangle$ to a hamiltonian cycle in $T_n\langle 1, 2; 2m\rangle$ by removing the edge $(n_0 - 1, n_0)$ and introducing the path

$$(n_0 - 1,\ n_0 + 1,\ n_0 + 2, \ldots, \underline{n - 1,\ n},\ n_0)$$

(see Figure 4).

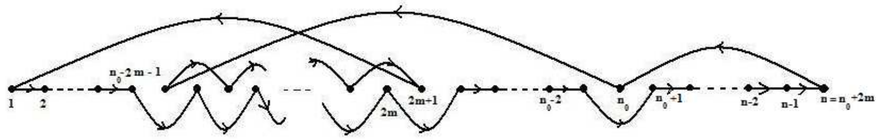

Fig. 4. $T_{n=n_0+2m}\langle 1,2;t_1\rangle$, for even t_1.

Suppose $T_n\langle 1,2;2m\rangle$ is hamiltonian for $n=n_0+q(2m)$ and for some positive integer q. We shall prove by induction on q that $T_n\langle 1,2;2m\rangle$ has a hamiltonian cycle containing the edge $(n-1,n)$. For $q=1$, the claim was verified above. Assume it is true for q, so $(n-1,n)$ is an edge in a hamiltonian cycle of $T_n\langle 1,\ 2;\ 2m\rangle$. Then we transform this cycle to a hamiltonian cycle in $T_{n+2m}\langle 1,\ 2;\ 2m\rangle$, by replacing the edge $(n-1,n)$ by the path

$$(n-1,n+1,n+2,\ldots,\underline{n+2m-1,n+2m},n).$$

Conversely, suppose H is a hamiltonian cycle in $T_n\langle 1,2;2m\rangle$. Then H is of course determined by the paths $H_{1\to n}$ and $H_{n\to 1}$.

Since $H_{1\to n}$ cannot use an increasing edge of length greater than 2, $H_{n\to 1}$ cannot use any increasing edge of the type $(v,v+1)$. Hence $H_{n\to 1}$ uses only edges of even length, and therefore vertices of the same parity, and this implies that n is odd. □

Theorem 3. *$T_n\langle 1,2;3\rangle$ is hamiltonian if and only if $n=5$ or $n\equiv 1\,(mod\,3)$.*

Proof. It is easily seen that $T_5\langle 1,2;3\rangle$ has the (unique) hamiltonian cycle

$$(1,\ 3,\ 5,\ 2,\ 4,\ 1).$$

Suppose now $n=1+3q$, $q\geq 1$. Then a hamiltonian cycle in $T_n\langle 1,2;3\rangle$ is

$$(1,2,3,5,6,8,9,\ldots,3q-1,3q,3q+1,3(q-1)+1,3(q-2)+1,\ldots,7,4,1)$$

(see Figure 5 for the case $n=25$).

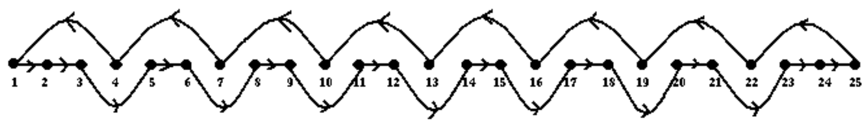

Fig. 5. $T_{25}\langle 1, 2; 3\rangle$.

Conversely, suppose $n \neq 5$ and $T_n\langle 1, 2; 3\rangle$ has a hamiltonian cycle H. Then $H_{n\to 1}$ uses edges of length 3 only, because no edge of $H_{1\to n}$ has length larger than 2. This implies that $n-1$ is a multiple of 3. □

Theorem 4. *$T_n\langle 1, 2; 5\rangle$ is hamiltonian for all $n \geqslant 29$.*

Proof. We take representatives 6, 9, 17, 25, 33 in the rest classes modulo 5.

For $n \in \{6, 9, 17, 25, 33\}$, $T_n\langle 1, 2; 5\rangle$ has the following hamiltonian cycle containing the edge $(n-1,\ n)$.

$n = 6$: (1, 2, 3, 4, 5, 6, 1),

$n = 9$: (1, 2, 3, 5, 7, 8, 9, 4, 6, 1),

$n = 17$:

(1, 2, 3, 5, 7, 8, 10, 11, 13, 15, 16, 17, 12, 14, 9, 4, 6, 1),

$n = 25$:

(1, 2, 3, 5, 7, 8, 10, 11, 13, 15, 16, 18, 19, 21, 23, 24, 25, 20, 22, 17, 12, 14, 9, 4, 6, 1),

$n = 33$:

(1, 2, 3, 5, 7, 8, 10, 11, 13, 15, 16, 18, 19, 21, 23, 24, 26, 27, 29, 31, 32, 33, 28, 30, 25, 20, 22, 17, 12, 14, 9, 4, 6, 1)

(for $n \in \{17, 25, 33\}$ see Figures 6-8).

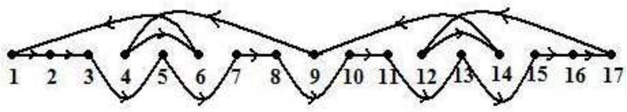

Fig. 6. $T_{17}\langle 1,2;5\rangle$.

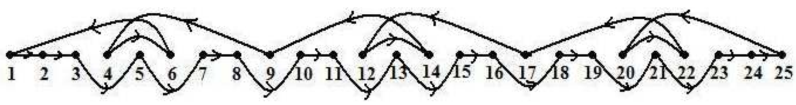

Fig. 7. $T_{25}\langle 1,2;5\rangle$.

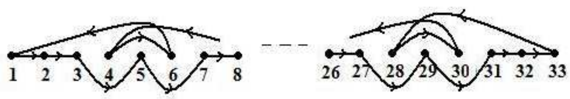

Fig. 8. $T_{33}\langle 1,2;5\rangle$.

Starting from the above values of n, we can extend a hamiltonian cycle in $T_n\langle 1,2;5\rangle$ containing the edge $(n-1,\, n)$ to a hamiltonian cycle in $T_{n+5}\langle 1,2;5\rangle$ with the same property by replacing the edge $(n-1,\, n)$ with the path

$$(n-1,\, n+1,\, n+2, ..., \underline{n+4,\, n+5},\, n).$$

Since $6, 17, 33, 9, 25$ are representatives in each of the various rest classes modulo 5, it follows that $T_n\langle 1,2;5\rangle$ is hamiltonian for all $n \geqslant 29$. □

We need the following technical lemmas for the proof of Theorem 5.

Lemma 1. *Let m be a positive odd integer, and n an integer larger than m. Then there exist non-negative integers q, r such that $n = 1 + qm - 2r$, q and n have opposite parity, and $r \le m-1$.*

Proof. By the division algorithm,

$$n = q_1 m + r_1; \quad 0 \le r_1 \le m-1.$$

Two cases arise:

Case 1 (n and q_1 have opposite parity)
In this case r_1 is odd.
If $r_1 = 1$, we write $n = 1 + q_1 m$.
If $r_1 > 1$, we write $n = 1 + (q_1 + 2)m - 2(m - (r_1 - 1)/2)$.

Case 2 (n and q_1 have same parity)
In this case r_1 is even and we write

$$n = 1 + (q_1 + 1)m - 2((m + 1 - r_1)/2).$$

□

Lemma 2. *Let $t_1 \geq 5$ be an odd integer. Then $T_n\langle 1, 2; t_1\rangle$ is hamiltonian if n can be written as*

$$n = 1 + qt_1 - 2r,$$

for some non-negative integers q and r, where $r \leqslant \frac{t_1 - 3}{2}\lfloor\frac{q}{2}\rfloor$.

Proof. First, write $r = r_1 + r_2 + \ldots + r_{\lfloor q/2\rfloor}$ with $r_i \leqslant \frac{t_1 - 3}{2}$. This is possible, because $r \leqslant \frac{t_1 - 3}{2}\lfloor\frac{q}{2}\rfloor$.

Claim: If $M \subset [1, n]$ includes $\{1, n\}$ and contains no pair of successive numbers, then there is a path $H_{1\to n}$ hamiltonian in the subgraph of $T_n\langle 1, 2; t_1\rangle$, spanned by

$$([1, n] \setminus M) \cup \{1, n\}.$$

This claim is rather obvious, as we have at our disposal increasing edges of both lengths 1 and 2.

We concentrate now on the proof of the existence of a path $H_{n\to 1} \subset T_n\langle 1, 2; t_1\rangle$ such that the set $M = V(H_{n\to 1})$ verifies the hypotheses of the claim (see Figure 9 for the case $t_1 = 9$ and $n = 25$).

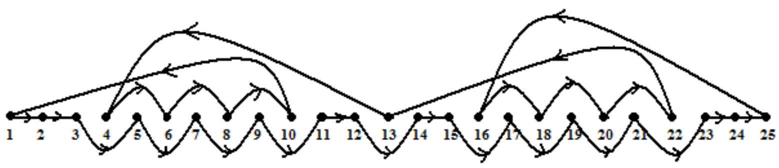

Fig. 9. $T_{25}\langle 1, 2; 9\rangle$, $M = \{1, 4, 6, 8, 10, 13, 16, 18, 20, 22, 25\}$, $r = 6$, $q = 4$.

With the claim in mind, it is clear that $H_{n\to 1}$ cannot use any increasing edge of length 1. So, it will use r increasing edges of length 2 and q decreasing edges of length t_1 in the following way. Between the first decreasing edge (of length t_1) and the second decreasing edge we insert $r_1 \leq (t_1-3)/2$ increasing edges of length 2. Between the third and the fourth decreasing edges we again insert $r_2 \leq (t_1-3)/2$ increasing edges of length 2. In general, between the $(2i-1)$-th and $2i$-th decreasing edges we insert $r_i \leq (t_1-3)/2$ increasing edges of length 2. Altogether $H_{n\to 1}$ uses q decreasing edges and $r = r_1 + r_2 + \ldots + r_{\lfloor q/2 \rfloor}$ increasing edges (of length 2).

This construction leads, after the use of all q decreasing edges and the intermediate $\sum r_i$ increasing edges to the final vertex 1, because

$$n - 1 = qt_1 - 2(r_1 + r_2 + \ldots + r_{\lfloor q/2 \rfloor}).$$

Since $V(H_{n\to 1})$ contains no pair of consecutive numbers, the claim yields a hamiltonian cycle $H_{1\to n} \cup H_{n\to 1}$ in $T_n\langle 1, 2; t_1\rangle$. □

Theorem 5. *Let $t_1 \geq 7$ be an odd integer. Then $T_n\langle 1, 2; t_1\rangle$ is hamiltonian for all $n > 3t_1 + 5$.*

Proof. As $n > t_1$, we can apply Lemma 1 and write $n = 1 + qt_1 - 2r$, where q and n have opposite parity. Also, by Lemma 2, $T_n\langle 1, 2; t_1\rangle$ is hamiltonian if $0 \leqslant r \leqslant \frac{t_1 - 3}{2}\lfloor \frac{q}{2} \rfloor$. We now verify these inequalities. Two cases arise as per parity of n.

Case 1. $n > 3t_1 + 5$ is even.

Indeed, for $n \in [3t_1 + 7,\ 5t_1 + 1]$, if we write

$$n = 1 + 5t_1 - 2r,$$

we have

$$r = \frac{1 + 5t_1 - n}{2} \geqslant 0$$

and $r = \frac{1 + 5t_1 - n}{2} \leqslant t_1 - 3$, because $n \geqslant 3t_1 + 7$;
so, $r \leqslant t_1 - 3 = \frac{t_1 - 3}{2}\lfloor \frac{5}{2} \rfloor$.

Now for $n \in [5t_1 + 3,\ 7t_1 + 1]$, if we write

$$n = 1 + 7t_1 - 2r,$$

we have

$$r = \frac{1 + 7t_1 - n}{2} \geqslant 0$$

and $r = \dfrac{1+7t_1-n}{2} \leqslant t_1 - 1$, because $n \geqslant 5t_1+3$;
so, $r \leqslant t_1 - 1 \leqslant (t_1-3)(\frac{3}{2}) = \dfrac{t_1-3}{2}\lfloor\dfrac{7}{2}\rfloor$.

Similarly for all $n \geqslant 7t_1+3$, we can write $n = 1 + qt_1 - 2r$, for some odd integer $q \geqslant 9$ and $0 \leqslant r \leqslant t_1 - 1 \leqslant \dfrac{t_1-3}{2}\lfloor\dfrac{q}{2}\rfloor$.

Case 2. $n > 3t_1 + 5$ is odd.

Indeed, for $n \in [3t_1+6,\, 4t_1+1]$, if we write

$$n = 1 + 4t_1 - 2r,$$

we have

$$r = \frac{1+4t_1-n}{2} \geqslant 0$$

and $r = \dfrac{1+4t_1-n}{2} \leqslant \dfrac{t_1-5}{2}$, because $n \geqslant 3t_1+6$;
so, $r \leqslant \dfrac{t_1-5}{2} < t_1 - 3 = \dfrac{t_1-3}{2}\lfloor\dfrac{4}{2}\rfloor$.

Now, for $n \in [4t_1+3,\, 6t_1+1]$, if we write

$$n = 1 + 6t_1 - 2r,$$

we have

$$r = \frac{1+6t_1-n}{2} \geqslant 0$$

and $r = \dfrac{1+6t_1-n}{2} \leqslant t_1 - 1$, because $n \geqslant 5t_1+3$;
so, $r \leqslant t_1 - 1 \leqslant (t_1-3)(\dfrac{3}{2}) = \dfrac{t_1-3}{2}\lfloor\dfrac{6}{2}\rfloor$.

Similarly for all $n \geqslant 6t_1+3$, we can write $n = 1 + qt_1 - 2r$, for some even integer $q \geqslant 8$ and $0 \leqslant r \leqslant t_1 - 1 \leqslant \dfrac{t_1-3}{2}\lfloor\dfrac{q}{2}\rfloor$.

Thus the conditions of Lemma 2 are verified in both subcases. This finishes the proof. □

3 Toeplitz graphs with $s_3 = 3$

In the previous section we saw that the hamiltonicity behaviour of $T_n\langle 1, 2; t_1\rangle$ strongly depends upon the parity of t_1 and n. In this section we require $s_3 = 3$. This strong requirement allows us to obtain the hamiltonicity of $T_n\langle 1, 2, 3; t_1\rangle$ for all $t_1 \geq 3$ and for all n (Theorem 7).

Among the Toeplitz graphs $T_n\langle 1, 2, 3; 1\rangle$, only $T_4\langle 1, 2, 3; 1\rangle$ is hamiltonian. For $t_1 = 2$, hamiltonicity is characterized in Theorem 6.

Theorem 6. *$T_n\langle 1, 2, 3; 2\rangle$ is hamiltonian if and only if $n = 4$ or $n \equiv 1\,(mod\,2)$.*

Proof. It is easily seen that $T_4\langle 1, 2, 3; 2\rangle$ has the (unique) hamiltonian cycle

$$(1,\ 4,\ 2,\ 3,\ 1).$$

Suppose now $n = 1+2q$, $q \geq 1$. Then a hamiltonian cycle in $T_n\langle 1, 2, 3; 2\rangle$ is

$$(1,\ 2,\ 4,\ 6, \ldots,\ n-1,\ n,\ n-2,\ n-4,\ n-6, \ldots,\ 3,\ 1)$$

(see Figure 10 for the case $n = 9$).

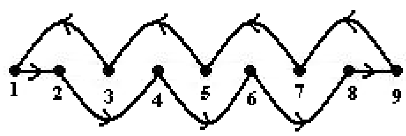

Fig. 10. $T_9\langle 1, 2, 3; 2\rangle$.

Conversely, suppose $n \neq 4$ and $T_n\langle 1, 2, 3; 2\rangle$ has a hamiltonian cycle H. Then $H_{n\to 1}$ uses edges of length 2 only, because no edge of $H_{1\to n}$ has length larger than 3. This implies that $n-1$ is a multiple of 2. □

Theorem 7. *$T_n\langle 1, 2, 3; t_1\rangle$ is hamiltonian for all $t_1 \geq 3$ and n.*

Proof.
Claim 1. For $n \in \{t_1{+}1,\ t_1{+}3,\ t_1{+}4, \ldots,\ 2t_1{-}1,\ 2t_1,\ 2t_1{+}2\}$, $T_n\langle 1, 2, 3; t_1\rangle$ has a hamiltonian cycle containing the edge $(n-1,\ n)$.

Indeed, $T_n\langle 1,2,3;n-1\rangle$ has the hamiltonian cycle $(1,\ 2,\ 3,\ldots,\ \underline{n-1,\ n},\ 1)$. $T_n\langle 1,2,3;t_1\rangle$ has, for $t_1+3 \le n \le 2t_1-2$, $t_1 \ge 5$, and n even, a hamiltonian cycle

$$(1,\ 2,\ 3,\ldots,\ n-t_1-1,\ n-t_1+1,\ n-t_1+2,\ n-t_1+4,\ n-t_1+6,\ldots,\\ t_1+2,\ t_1+3,\ldots,\underline{n-1,\ n},\ n-t_1,\ n-t_1+3,\ n-t_1+5,\ldots,\ t_1+1,\ 1)$$

(see Figure 11).

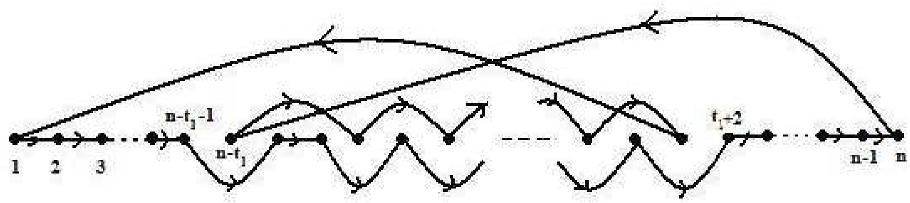

Fig. 11. $T_n\langle 1,2,3;t_1\rangle$; $t_1+3 \le n \le 2t_1-2$ and n is even.

$T_n\langle 1,2,3;t_1\rangle$ has, for $t_1+3 \le n \le 2t_1-1$, $t_1 \ge 4$, and n odd, a hamiltonian cycle

$$(1,\ 2,\ 3,\ldots,\ n-t_1-1,\ n-t_1+1,\ n-t_1+3,\ n-t_1+5,\ldots,t_1+2,\\ t_1+3,\ldots,\underline{n-1,\ n},\ n-t_1,\ n-t_1+2,\ n-t_1+4,\ldots,\ t_1+1,\ 1)$$

(see Figure 12).

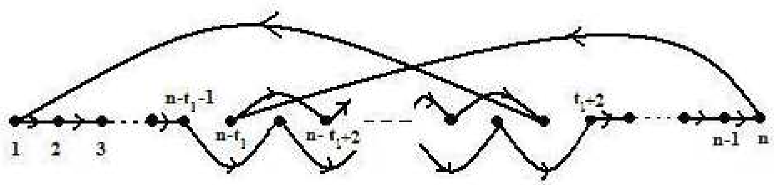

Fig. 12. $T_n\langle 1,2,3;t_1\rangle$; $t_1+3 \le n \le 2t_1-1$ and n is odd.

$T_{2t_1}\langle 1, 2, 3; t_1\rangle$ has a hamiltonian cycle

$$(1,\ 2,\ 3, \ldots,\ n-t_1-1,\ n-t_1+2 = t_1+2,\ t_1+3, \ldots,\ \underline{n-1,\ n},\ n-t_1 = t_1,\ t_1+1,\ 1)$$

(see Figure 13).

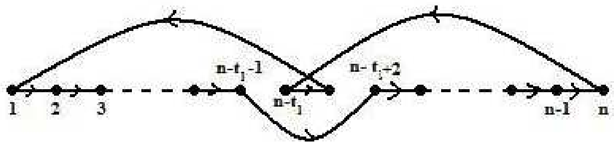

Fig. 13. $T_{2t_1}\langle 1, 2, 3; t_1\rangle$.

For $T_{2t_1+2}\langle 1, 2, 3; t_1\rangle$, we have the following two subcases as per parity of t_1.

(a) If t_1 is odd, then a hamiltonian cycle in $T_n\langle 1, 2, 3; t_1\rangle$ is

$$(1,\ 3,\ 5, \ldots,\ t_1,\ t_1+3,\ t_1+4, \ldots,\ \underline{n-1,\ n},\ n-t_1,\ 2,\ 4,\ 6, \ldots, t_1+1,\ 1)$$

(see Figure 14).

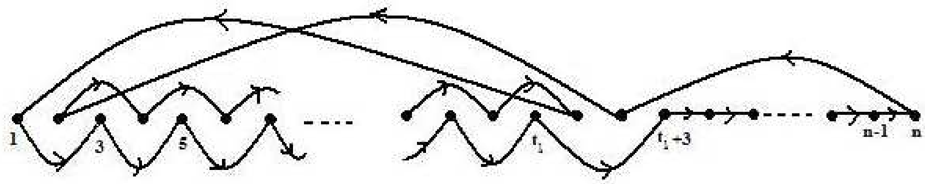

Fig. 14. $T_{2t_1+2}\langle 1, 2, 3; t_1\rangle$; t_1 is odd.

(b) If t_1 is even, then a hamiltonian cycle in $T_n\langle 1, 2, 3; t_1\rangle$ is

$$(1,\ 3,\ 4,\ 6,\ 8, \ldots,\ t_1,\ t_1+3,\ t_1+4, \ldots,\ \underline{n-1,\ n},\ n-t_1,\ 2,\ 5,\ 7,\ 9, \ldots,\ t_1+1,\ 1)$$

(see Figure 15).

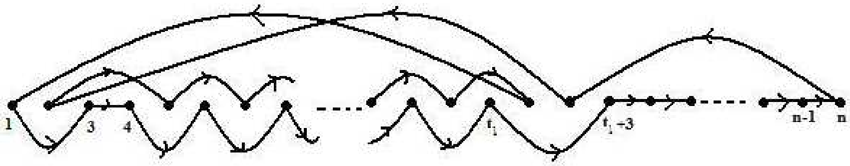

Fig. 15. $T_{2t_1+2}\langle 1, 2, 3; t_1\rangle$; t_1 is even.

Claim 2. $T_{t_1+2}\langle 1, 2, 3; t_1\rangle$ is hamiltonian.

Indeed, we have the following two cases as per parity of t_1.
(a) If t_1 is odd, then a hamiltonian cycle in $T_{t_1+2}\langle 1, 2, 3; t_1\rangle$ is

$$(1,\, 3,\, 5, \ldots,\, t_1 + 2 = n,\, 2,\, 4,\, 6, \ldots,\, t_1 + 1 = n - 1,\, 1)$$

(see Figure 16).

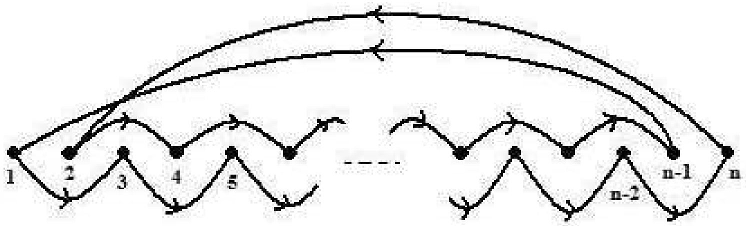

Fig. 16. $T_{t_1+2}\langle 1, 2, 3; t_1\rangle$; t_1 is odd.

(b) If t_1 is even, then a hamiltonian cycle in $T_{t_1+2}\langle 1, 2, 3; t_1\rangle$ is

$$(1,\, 3,\, 4,\, 6,\, 8, \ldots,\, t_1,\, t_1 + 2 = n,\, 2,\, 5,\, 7, \ldots,\, t_1 + 1 = n - 1,\, 1)$$

(see Figure 17).

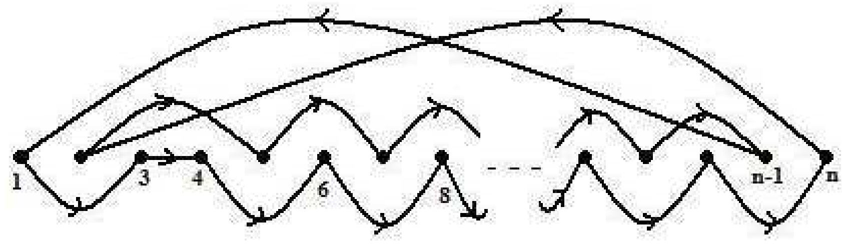

Fig. 17. $T_{t_1+2}\langle 1, 2, 3; t_1\rangle$; t_1 is even.

Suppose $T_n\langle 1, 2, 3; t_1\rangle$ has, for $n \neq t_1 + 2$ and $n = n_0 + qt_1$, a hamiltonian cycle containing the edge $(n-1,\, n)$, for some non-negative integer q. We shall prove that $T_{n+t_1}\langle 1, 2, 3; t_1\rangle$ has the same property.

Since $(n-1,\, n)$ is an edge in a hamiltonian cycle of $T_n\langle 1, 2, 3; t_1\rangle$, we transform this cycle to a hamiltonian cycle in
$T_{n+t_1}\langle 1, 2, 3; t_1\rangle$ by replacing the edge $(n-1,\, n)$ with the path

$$(n-1,\, n+1,\, n+2, \ldots,\, \underline{n+t_1-1,\, n+t_1},\, n).$$

By Claim 1, $T_n\langle 1, 2, 3; t_1\rangle$ enjoys the above property for $n \in \{t_1+1,\, t_1+3,\, t_1+4, \ldots,\, 2t_1-1,\, 2t_1,\, 2t_1+2\}$. It follows that the property holds for $n = t_1 + 1$ and all $n \geq t_1 + 3$. This together with Claim 2 shows that $T_n\langle 1, 2, 3; t_1\rangle$ is hamiltonian for all n. □

4 Concluding Remarks

The investigation of the hamiltonicity of Toeplitz graphs, directed or not, is far from being achieved. We studied in this paper only the cases of small numbers k, l and s_i, t_j. Of course, these cases are most relevant to this study, but there is still much left to do. For $s_2 = 2$ and $s_3 = 3$ we may consider the investigation as complete. The next task is, in our opinion, the investigation of the case of k, l still small, but larger s_i, t_j.

Also, other characterizations of hamiltonian graphs inside subfamilies of Toeplitz graphs would be most welcome.

In this paper we provided no negative results, except for those implied by the characterizations of hamiltonian graphs inside classes of Toeplitz graphs. Such results, besides those in [3] yielding disconnectedness, would also be of interest.

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