Hamiltonicity of cubic 3-connected k-Halin graphs

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Abstract

We investigate here how far we can extend the notion of a Halin graph such that hamiltonicity is preserved. Let $H = T \cup C$ be a Halin graph, T being a tree and C the outer cycle. A k-Halin graph G can be obtained from H by adding edges while keeping planarity, joining vertices of $H - C$, such that $G - C$ has at most k cycles. We prove that, in the class of cubic 3-connected graphs, all 14-Halin graphs are hamiltonian and all 7-Halin graphs are 1-edge hamiltonian. These results are best possible.

Keywords: Halin graph, k-Halin graph, hamiltonian, k-edge hamiltonian.

1 Introduction

A *Halin graph* is a plane graph $G = T \cup C$, where $T \neq K_2$ is a tree with no vertex of degree two, and C is a cycle connecting the leaves of T in the cyclic order determined by a plane embedding of T. Halin graphs belong to the family of all planar, 3-connected graphs and possess strong hamiltonian properties. We presented in [13] the following generalization of Halin graphs.

A 2-connected planar graph G without vertices of degree 2, possessing a cycle C such that
(i) all vertices of C have degree 3 in G, and
(ii) $G - C$ is connected and has at most k cycles
is called a *k-Halin graph*. The cycle C is called the *outer cycle*, while the cycles in $G - C$ are called *inner cycles* of G.

A graph on n vertices is called

- 1-*hamiltonian*, if it is hamiltonian and remains so after removal of an arbitrary vertex.
- *hamiltonian connected*, if any pair of (distinct) vertices is connected by a hamiltonian path.
- k-*edge hamiltonian*, if every disjoint union of paths (of positive length each), of total length k, lies in some hamiltonian cycle.
- *pancyclic*, if it contains cycles of all lengths from 3 to n.

0-*Halin* graphs are usual *Halin* graphs. They are 1-hamiltonian and hamiltonian connected, as proved by Bondy (see [12]), and Barefoot [1], respectively. As expected, as k increases, the hamiltonicity of k-Halin graphs steadily decreases. It was proved in [13] that a 1-Halin graph is still hamiltonian, but may not be hamiltonian connected, a 2-Halin graph is not necessarily hamiltonian but still traceable, while a 3-Halin graph is not even necessarily traceable. However, like Halin graphs, those 3-Halin graphs, in which vertices of degree 3 are on the outer cycle only, are pancyclic.

For k-Halin graphs the property of being hamiltonian persists, nevertheless, for large values of k in the family $\mathcal{G}$ of all planar, cubic 3-connected graphs. We prove here that every 14-Halin graph in $\mathcal{G}$ is hamiltonian. A variant of the famous example of Tutte [19] from 1946, which first demonstrated that graphs in $\mathcal{G}$ may not be hamiltonian, is a 21-Halin graph. The cubic 3-connected planar non-hamiltonian graph of Lederberg [10], Bosák [5] and Barnette (see [2]), which has smallest order, is 53-Halin. We show here that there exist non-hamiltonian 15-Halin graphs in $\mathcal{G}$. 7-Halin graphs from $\mathcal{G}$ are even 1-edge hamiltonian, as we shall prove. This result, too, is best possible.

We chose to use in the definition of a k-Halin graph the number of all inner cycles, not only of those bounding faces, because the results of this paper can be better formulated this way.

Other generalizations of Halin graphs and investigation of their hamiltonian properties have already been published by Skowrońska [16], Skowrońska and Sysło [17], Skupień [18], Zamfirescu and Zamfirescu [20].

2 Two-edge hamiltonicity

Every Halin graph is 1-edge hamiltonian. This follows from the hamiltonian connectivity, and from the 1-edge hamiltonicity of graphs more general than Halin graphs, proved by Skowrońska [16]. We start with the following result on Halin graphs, which seems to be new.

Theorem 1. *Every cubic Halin graph is 2-edge hamiltonian.*

Proof. The statement is verified for K_4, the smallest Halin graph. Let H be a cubic Halin graph on more than 4 vertices. We assume the statement true for all Halin graphs of smaller order, and prove it for H.

Let C be the outer cycle of H. Let a, a' be the end points of a longest path in the tree $H - C$. Let b, c be the neighbours of a on C. Consider the path $b'bcc' \subset C$. If we remove b and c from H and add the edges (a, b') and (a, c'), we obtain a smaller Halin graph H_a. Let e, e' be two edges of H.

If e, e' lies in H_a, then they lie on some hamiltonian cycle in H_a. This cycle can be modified to become a hamiltonian cycle in H by replacing ab' with $acbb'$ or ae' with $abce'$.

If e lies in H_a and $e' \in (a, c), (b, c), (b, b')$ then we chose a hamiltonian cycle in H_a containing e and (a, b'), and replace afterwards ab' by $acbb'$.

If e lies in H_a and $e' \in (a, b), (c, a')$, then we chose a hamiltonian cycle in H_a containing e and (a, c'), and then replace ac' by $abca'$.

If neither e, nor e', lies in H_a, then both e, e' lie in H'_a and we proceed analogously with H'_a instead of H_a. □

The graph obtained from a Halin graph by deleting a vertex x of its outer cycle is called a *reduced Halin graph* [6]. The three neighbouring vertices of x, whose degrees reduce by one, are called the *end-vertices* of the reduced Halin graph. Lemma 1 of [6] tells us the following.

Lemma 1. *In any reduced Halin graph each pair of end-vertices is joined by a hamiltonian path.*

This is a consequence of the uniform hamiltonicity of Halin graphs, proved by Skupień [18]. A graph is called *uniformly hamiltonian* if it is 1-edge hamiltonian and each edge is missed by some hamiltonian cycle. As a consequence of Theorem 1 we obtain the following strengthening for graphs in $\mathcal{G}$.

Corollary 1. *In any reduced Halin graph belonging to $\mathcal{G}$, for any edge e and end-vertex v, there is a hamiltonian path containing e and joining v to another end-vertex.*

From Corollary 1 we easily deduce the following.

Corollary 2. *In any reduced Halin graph belonging to $\mathcal{G}$, for any edge e there is an end-vertex v, such that e lies on two hamiltonian paths, one joining v to another end-vertex, and the other joining v to the third end-vertex.*

By Lemma 1, we can contract any reduced Halin subgraph of a graph G to a single vertex of degree 3, without changing the hamiltonicity (or non-hamiltonicity) of G.

3 14-Halin graphs with connected core

In a k-Halin graph ($k \geqslant 1$), the union of all inner cycles will be called its *core*.

Lederberg [10], Bosák [5] and Barnette (see [2]) found independently in 1966 the planar cubic 3-connected non-hamiltonian graph with 38 vertices mentioned in the Introduction.

This was not the first non-hamiltonian graph found in $\mathcal{G}$, but rather small. At the end of a sequence of papers by Lederberg [11], Butler [7], Barnette and Wegner [3], Okamura [14], [15], again Barnette [4], Holton and McKay [9], which appeared between 1966 and 1989, it was eventually proven that no smaller examples exist. This is stated in the next lemma.

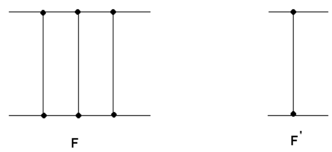

Figure 1:

Lemma 2. *All graphs in $\mathcal{G}$ with at most 36 vertices are hamiltonian.*

The following lemma is straightforward.

Lemma 3. *Let G contain the fragment F shown in Figure 1. Replace F with the fragment F' and obtain a graph G'. If G' is hamiltonian then G is also hamiltonian.*

Theorem 2. *Every cubic 3-connected 14-Halin graph with a connected core is hamiltonian.*

Proof. Let G have k inner cycles ($k \leqslant 14$) and denote by K the core of G. Then K may only take one of the forms shown in Figure 2.

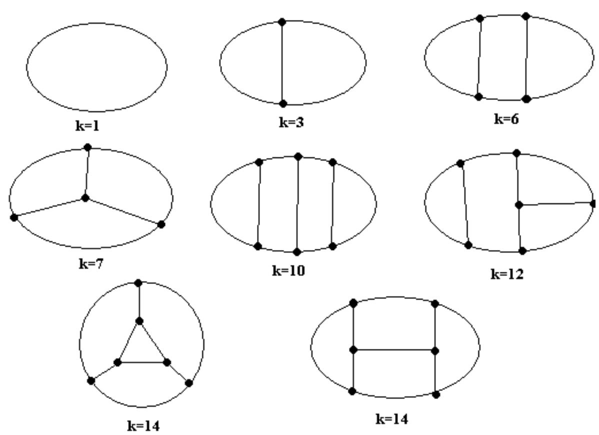

Figure 2:

There are 7 possible values for k, namely $1, 3, 6, 7, 10, 12, 14$. Our graph G is the union of K with the outer cycle C and with trees having all vertices of degree 3, except for the leaves, one of which belongs to K and all the others to C. By Lemma 1, each such tree may be contracted to an edge between K and C without affecting the hamiltonicity of

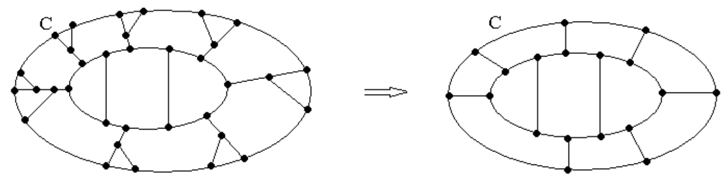

Figure 3:

G. This transforms G into a simple 14-Halin graph (Figure 3). (For the definition of a simple k-Halin graph, see the beginning of Section 3.)

Without gaining hamiltonicity, we may further reduce the order of G by observing that a repeated use of Lemma 3 decreases the number of consecutive edges between K and C to at most two. Here, "consecutive edges"means that their vertices in K are adjacent. We obtain a graph as depicted in Figure 4, where in each case the maximal number, two, of consecutive edges is shown. In every case the order is at most 30. So, by Lemma 2, the graph is hamiltonian. □

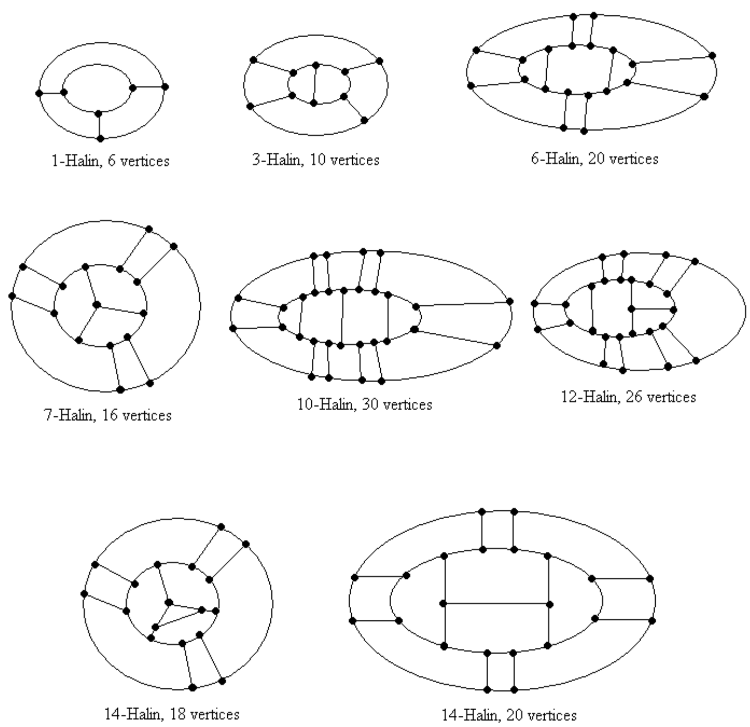

Figure 4:

4 Almost k-Halin graphs

Let $k \geqslant 1$. A path in a k-Halin graph is called *inner*, if it has its end-vertices on distinct inner cycles and no other vertex on any inner or outer cycle.

A k-Halin graph ($k \geqslant 1$) is called *simple* if it is spanned by the union of all its inner paths and cycles and the outer cycle. We say that a graph is *almost k-Halin* if it is obtained from a simple cubic 3-connected k-Halin graph, which is not $(k-1)$-Halin and has a connected core, by deleting a vertex of its outer cycle. The neighbours of that vertex are called the *end-vertices* of the almost k-Halin graph.

Lemma 4. *In any almost* 1*-Halin graph each pair of end-vertices is joined by a hamiltonian path.*

Proof. Let F be an almost 1-Halin graph with end-vertices b_1, b_2, b_3. By definition, there exists a 1-Halin graph $G \in \mathcal{G}$ and a vertex a on its outer cycle C, with neighbours b_1, b_2, b_3, such that $F = G - a$ (Figure 5).

It is easy to see that G has a hamiltonian cycle containing the path b_1ab_2 on its outer cycle. By deleting a, we get a hamiltonian path between b_1 and b_2 in F. Similarly we can find hamiltonian paths between b_1, b_3 and b_2, b_3 in F. $\square$

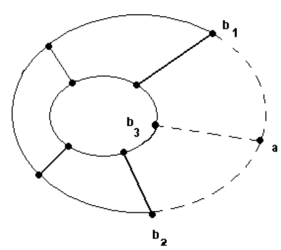

Figure 5:

Lemma 5. *In any almost* 3*-Halin graph, each pair of end-vertices is joined by a hamiltonian path.*

Proof. The almost 3-Halin graph F defines as in the preceding proof a graph G and a vertex $a \in G$ with the end-vertices b_1, b_2, b_3 as neighbours. We use the notation of Figure 6. We apply Lemma 3 to reduce the number of consecutive edges between C and the core.
Case 1. We have only one edge between C_1 and C.

In this case C_1 is a triangle and therefore may be contracted to a vertex of degree 3 without affecting the hamiltonicity and end-vertices of F. This reduces F to an almost 1-Halin graph, which has, by Lemma 4, a hamiltonian path between each pair of end-vertices.

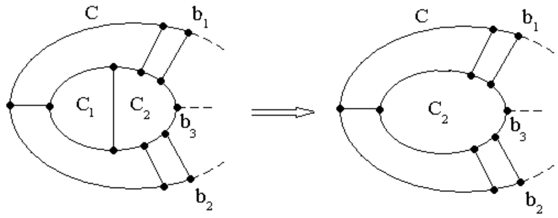

Figure 6:

Case 2. There are exactly two edges between C_1 and C.

In this case C_1 along with the pair of consecutive edges between it and C may be contracted, by Lemma 3, to a single edge, as shown in Figure 7.

Thus, the original graph becomes a reduced Halin graph, which has, by Lemma 1, a hamiltonian path between each pair of end-vertices. □

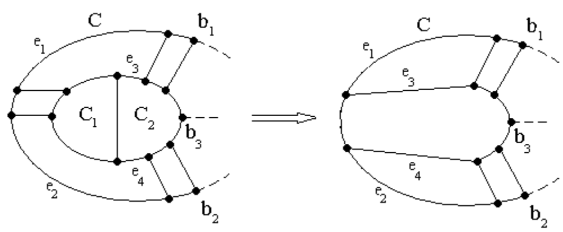

Figure 7:

Lemma 6. *In any almost* 6*-Halin graph, each pair of end-vertices is joined by a hamiltonian path.*

Proof. We use the same notation. Let C_i $(i = 1, 2, 3)$ be cycles of the core, as shown in Figure 8.

After reducing the number of consecutive edges between C and the core, we have to consider two essentially different cases for F, according to the position of b_3 in the core.

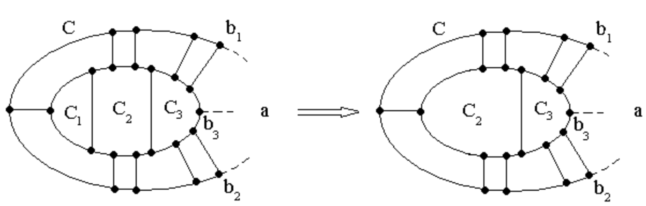

Figure 8:

Case 1. The end-vertex b_3 belongs to C_3.

We have two subcases.

(i) There is one edge between C_1 and C.

In this case, the triangle C_1 may be contracted to a single vertex, which transforms the graph to an almost 3-Halin graph (see Figure 8). By Lemma 5, the latter has a hamiltonian path between each pair of end-vertices.

(ii) There are two edges between C_1 and C.

As in Case 2 of Lemma 5, the graph may be contracted to the graph shown in Figure 9 (middle). Keeping hamiltonicity, we can successively contract a triangle to a vertex until the graph becomes an almost 1-Halin graph (see Figure 9). By Lemma 4, this has a hamiltonian path between each pair of end-vertices.

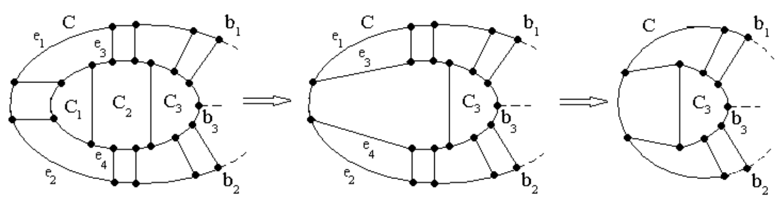

Figure 9:

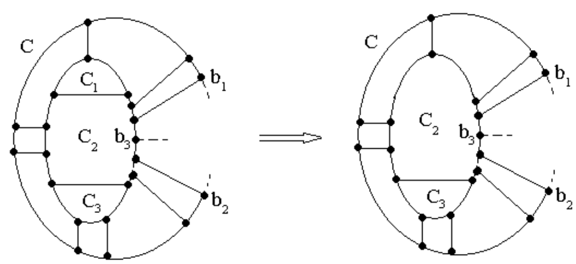

Figure 10:

Case 2. The end-vertex b_3 belongs to C_2.

There are two possibilities.

(i) There is one edge between C_1 and C.

Then the graph may be contracted to an almost 3-Halin graph (see Figure 10) which, by Lemma 5, has a hamiltonian path between each pair of end-vertices.

(ii) There are two edges between C and C_1.

The graph may be contracted to the graphs shown in succession in Figure 11.

A hamiltonian path between any pair of end-vertices depends upon the number x of edges at place X in the graph. This directs us to consider the following cases.

(i) $x = 1$.

Hamiltonian paths between end-vertices are shown in Figure 12.

(ii) $x = 2$ (or 0).

Hamiltonian paths between end-vertices are shown in Figure 13. $\square$

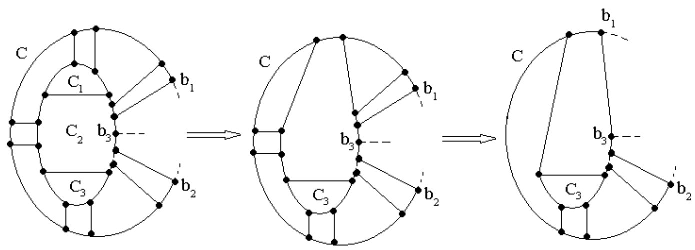

Figure 11:

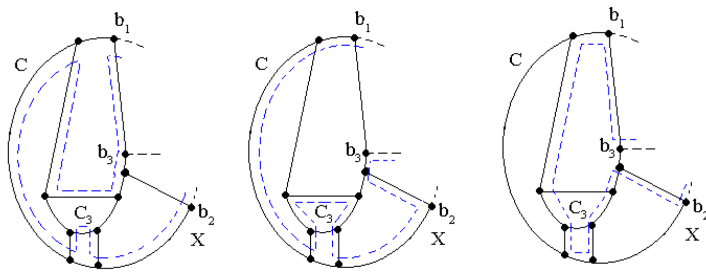

Figure 12:

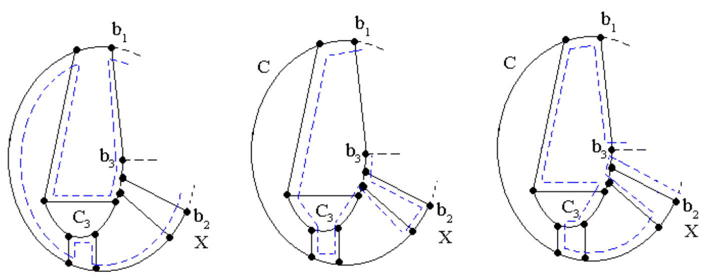

Figure 13:

Lemma 7. *In any almost* 10*-Halin graph, each pair of end-vertices is joined by a hamiltonian path.*

Proof. We continue to use the previous notation. Let C_i $(i = 1, 2, 3, 4)$ be cycles of the core, as shown in Figure 14.

After reduction of consecutive edges we are led to consider two essentially different cases for F.

Case 1. The end-vertex b_3 belongs to C_4.

We have two subcases to consider.

(i) There is one edge between C_1 and C.

In this case the graph may be contracted to an almost 6-Halin graph (Figure 15). We apply Lemma 6.

(ii) There are two edges between C_1 and C.

We contract the graph to an almost 3-Halin graph (see Figure 16) and apply Lemma 5.

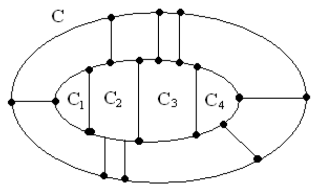

Figure 14:

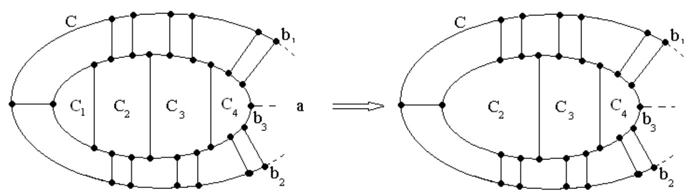

Figure 15:

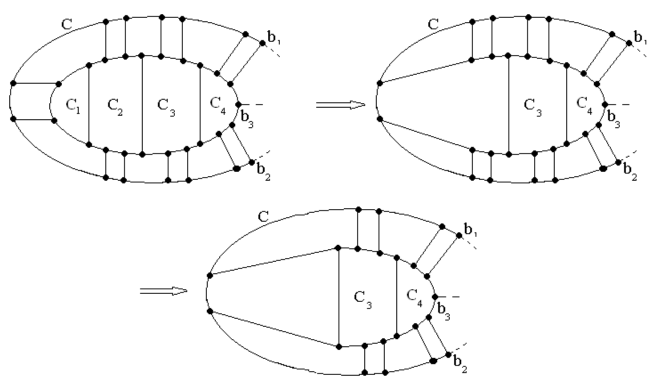

Figure 16:

Case 2. The end-vertex b_3 belongs to C_3.

We have two subcases to consider.

(i) There is one edge between C_1 and C.

The graph may be contracted to an almost 6-Halin graph (Figure 17).

(ii) There are two edges between C and C_1.

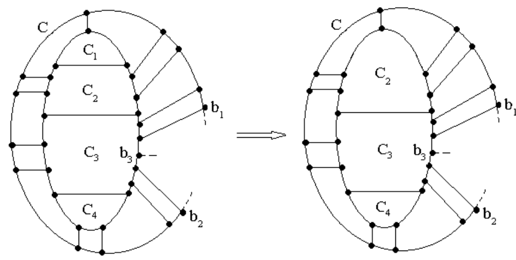

Figure 17:

Figure 18 shows how the graph may be contracted to an almost 3-Halin graph. $\square$

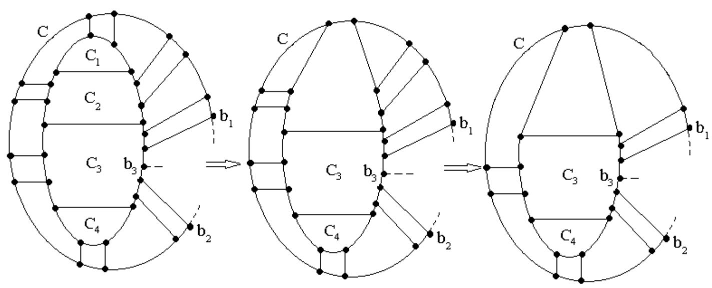

Figure 18:

Lemmas 4, 5, 6 and 7 allow us to contract any almost k-Halin graph ($k \in \{1, 3, 6, 10\}$) to a vertex of degree 3, just as we can do with reduced Halin graphs.

Lemma 8. *In any almost* 7*-Halin graph, at least two pairs of end-vertices are joined by hamiltonian paths.*

Proof. Following the notation of Lemma 4 and using Lemma 3, F will have at most 21 vertices, as shown in Figure 19.

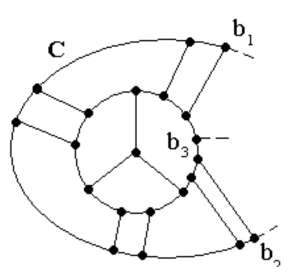

Figure 19:

Assume F has only one pair of end-vertices, say b_1, b_2, joined by a hamiltonian path. It is well known that the Tutte triangle, i.e., the graph of Figure 20, has no hamiltonian path from u to v [19]. Then, by joining the end-vertices of F with the vertices of degree 2 of the Tutte triangle appropriately (see Figure 21), we obtain a plane cubic 3-connected non-hamiltonian graph on at most $21 + 15 = 36$ vertices. This contradicts Lemma 2. Hence F must have at least two pairs of end-vertices joined by hamiltonian paths. □

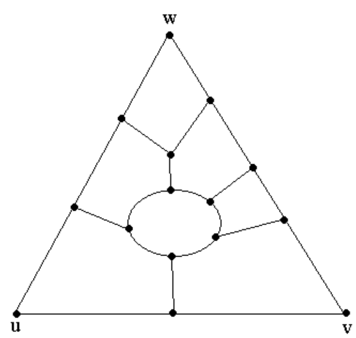

Figure 20:

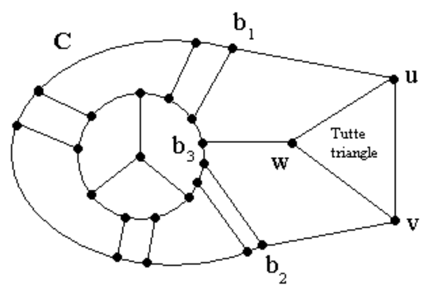

Figure 21:

5 14-Halin graphs

Theorem 3. *Let G be a cubic 3-connected k-Halin graph with core K, such that one component of K has at most 14 cycles and the number of cycles in any other component of K lies in $\{1, 3, 6, 10\}$. Then G is hamiltonian.*

Proof. First contract each tree with one leave a on an inner path or cycle and all the others on the outer cycle C to an edge from a to C. So, G becomes simple. The graph having the components of K as vertices and the inner paths of G as edges between the corresponding vertices is a tree T.

Let x be a leave of T not corresponding to the component of K with at most 14 cycles mentioned in the statement. Thus, x corresponds to a component of K with $1, 3, 6$ or 10 cycles. By Lemmas $4, 5, 6, 7$, that component may be contracted to a vertex, and for the new graph the tree T has one vertex less. By continuing this procedure, we arrive at a simple 14-Halin graph with connected core, which is by Theorem 2 hamiltonian. □

Theorem 4. *Every cubic* 3*-connected* 14*-Halin graph is hamiltonian.*

Proof. Let G be a cubic 3-connected 14-Halin graph. We use the notation of Theorem 3. If the components of K satisfy the requirements of Theorem 3, then G is hamiltonian. If not, then K must have exactly two components, each with 7 cycles. We first reduce G into a simple 14-Halin graph using Lemma 1. By Lemma 8, an almost 7-Halin graph has at least two pairs of end-vertices joined by a hamiltonian path. Let us call such pairs *workable*. G looks as shown in Figure 22 and is the union of two almost 7-Halin graphs H, H' and a subgraph L depicted in Figure 23.

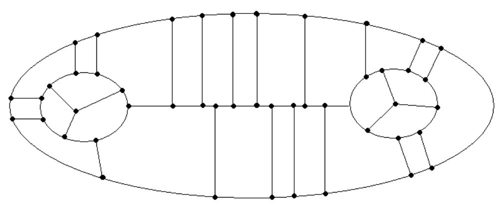

Figure 22:

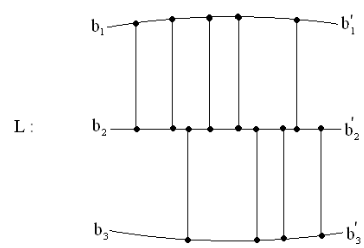

Figure 23:

From any pair of end-vertices b_1, b_2 and b_3 of H the subgraph L is traversed in a unique way, leading to a unique pair of end-vertices b_1', b_2', b_3' of H'. This obviously establishes a one-to-one correspondence between the pairs (b_1, b_2), (b_2, b_3), (b_3, b_1) and the pairs (b_1', b_2'), (b_2', b_3'), (b_3', b_1') (see Figure 24.)

Since there are at least two workable pairs of end-vertices on each side, at least one workable pair on one side is in correspondence with a workable pair on the other side. This guarantees the existence of a hamiltonian cycle in G. □

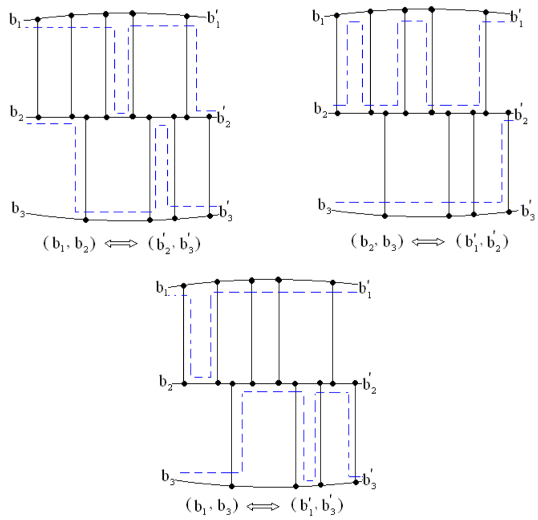

Figure 24:

Theorem 5. *There are cubic* 3*-connected* 15*-Halin graphs which are not hamiltonian.*

Proof. Figure 25 depicts a cubic 3-connected 15-Halin graph G on 42 vertices that comprises two Tutte triangles T, T', each containing a component of the core with 7 cycles, and one additional component with one cycle in the middle. Since there is no hamiltonian path between a and b in T, the workable pairs in T are a, c and b, c only. And the workable pairs in T' are a', c' and b', c'.

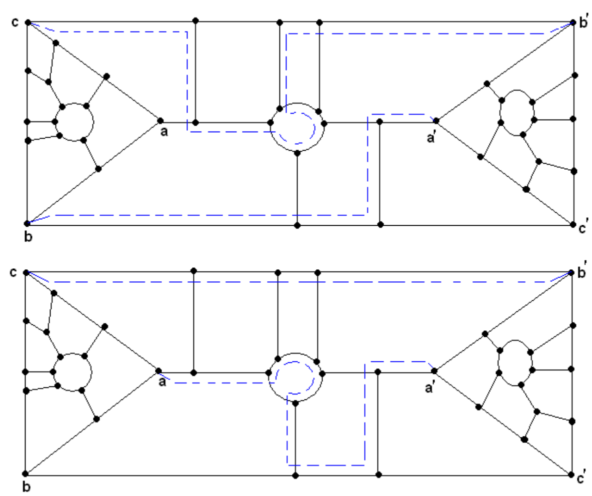

Figure 25:

Figure 25 also shows that any pair of disjoint paths from a workable pair of T goes through the middle inner cycle in a unique way and eventually ends at the pair a', b' of T'. Hence G is not hamiltonian. □

6 1-edge hamiltonicity of 7-Halin graphs

By using computers it was determined that the graph of Figure 26 has smallest order, 24, among all graphs in $\mathcal{G}$ which are not 1-edge hamiltonian [8]. More precisely, each of the edges e and e' belongs to no hamiltonian cycle. This is a 10-Halin graph. Are all 9-Halin graphs in $\mathcal{G}$ 1-edge hamiltonian?

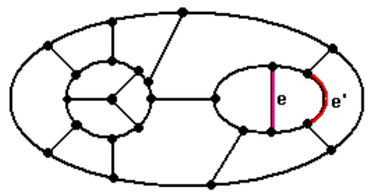

Figure 26:

Theorem 6. *Every cubic* 3*-connected* 7*-Halin graph is* 1*-edge hamiltonian.*

Proof. Let $G \in \mathcal{G}$ be a 7-Halin graph. Let K be its core and C its outer cycle. Let $e \in E(G)$. We consider the following two cases.

Case 1. K is connected.

Using reduction techniques based on Lemmas 1 and 3, as in Theorem 2, G becomes a graph of order at most 20 (see Fig. 4). Suppose now that e belonged to one of the maximal reduced Halin graphs, say H, which was contracted to the vertex $v \in C$. Let $a, b \in C$ and c be the end-vertices of H. If a', b' are the vertices to which the maximal reduced Halin graphs containing a, b were contracted, then $a', b', c = c'$ are the neighbours of v. Assume, for example, that a is the end-vertex of H provided by Corollary 2. Then after contraction, e is replaced by (v, a'). Since all graphs in $\mathcal{G}$ on less than 24 vertices are 1-edge hamiltonian, G has a hamiltonian cycle containing e.

Case 2. K is disconnected.

By Lemma 1, we contract any maximal reduced Halin subgraph in G to a vertex of degree 3 on C. Lemmas 4, 5, 6 further allow us to contract any almost Halin graph in G that does not contain e, to a vertex of degree 3 on C, proceeding like in the proof of Theorem 3. This reduces G to a k-Halin graph ($k < 7$) with connected core, which by Case 1 possesses a hamiltonian cycle containing e. $\square$

Theorem 7. *There are cubic* 3*-connected* 8*-Halin graphs which are not* 1*-edge hamiltonian.*

Proof. Figure 27 depicts an 8-Halin graph $G \in \mathcal{G}$ on 26 vertices that contains a Tutte triangle T, containing a component of the core with 7 cycles, and one additional component with one cycle. Since there is no hamiltonian path between b and c in T, the workable pairs in T are c, a and a, b only.

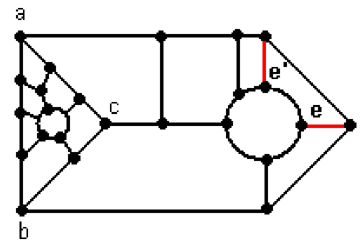

Figure 27:

Figure 28 shows that any pair of disjoint paths from a workable pair of T goes through $G - T$ in a unique way that avoids both edges e and e'. Accordingly, G is not 1-edge hamiltonian. $\square$

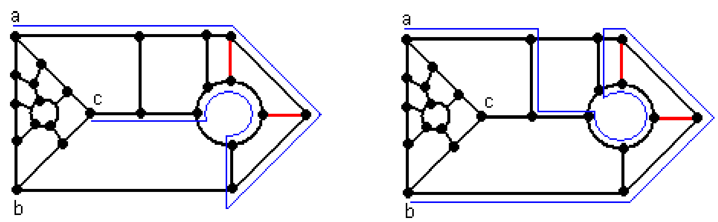

Figure 28:

7 Final remarks

Theorems 4 and 5 settle the problem of finding the maximal k for which all cubic 3-connected k-Halin graphs are hamiltonian. It would be interesting to investigate the analogous question about traceability. However, the exact calculation of the maximal number k up to which all cubic 3-connected k-Halin graphs are traceable seems out of reach, as the corresponding, more general problem of finding the maximal order up to which all cubic 3-connected graphs are traceable is very far from being solved.

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