Hamiltonicity in Directed Toeplitz Graphs of Maximum (out or in)Degree 4

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Abstract. A directed Toeplitz graph is a digraph with a Toeplitz adjacency matrix. In this paper we study the hamiltonicity of the Toeplitz graphs of type $T_n\langle 1, 3, 4; t\rangle$. For $t \in \{2, 3, 4, 5, 8, 9\}$, we give conditions (on n) under which such a graph is hamiltonian. For $t \in \{6, 7\}$ and $t \geq 10$, we see that $T_n\langle 1, 3, 4; t\rangle$ is hamiltonian for all n.

Keywords: *Toeplitz graph; Hamiltonian graph.*

1 Introduction

In this paper all graphs are directed. For a digraph (directed graph) D, as usual, $V(D)$ will denote its vertex set and $A(G)$ its arc (directed edge) set. A digraph C with $V(C) = \{v_1, \ldots, v_n\}$ and $A(C)$=$\{(v_1, v_2)$, (v_2, v_3),. . . ,(v_{n-1}, v_n),$(v_n, v_1)\}$ is called a *circuit* (Of course, $v_i \neq v_j$ for all distinct i, j). A circuit minus one arc is called a *path*. A digraph D' is called a *sub(di)graph* of D if $V(D') \subset V(D)$ and $A(D') \subset A(D)$. If moreover $V(D') = V(D)$, D' is said to *span* D. If D' spans D and is a circuit or a path, it is also called *hamiltonian*. Any digraph possessing a hamiltonian circuit is itself called *hamiltonian*, too. Indegree (outdegree) of a vertex v in D is the number of head (tail) endpoints adjacent to v and is denoted by $d^-(v)$ $(d^+(v))$. An arc (v_1, v_2) is increasing (decreasing) if $(v_1 < v_2)$ $((v_1 > v_2)$, respectively).

The directed Toeplitz graph $T_n\langle\, s_1,\, s_2,\ldots,\, s_k;\, t_1,\, t_2,\ldots, t_l\rangle$ is the digraph with vertices 1, 2, . . . , n, in which the arc (i, j) occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some integers p and q $(1 \leq p \leq k,\ 1 \leq q \leq l)$. Its adjacency matrix is a Toeplitz matrix, i.e., it has constant values along all diagonals parallel to the main diagonal. If the Toeplitz adjacency matrix is symmetric, the graph is said to be undirected. $T_n\langle t_1, t_2, \ldots, t_i\rangle$ denotes the undirected Toeplitz graph with the adjacency matrix of $T_n\langle\, t_1,$

$\ldots,\ t_i;\ t_1,\ \ldots,\ t_i\rangle$. Hamiltonicity of $T_n\langle\ t_1,t_2,\ldots,t_i\rangle$ means hamiltonicity of $T_n\langle\ t_1,\ldots,t_i;t_1,\ldots\ ,t_i\rangle$. Connectivity results obtained in the undirected case have a direct impact on the directed case. So connectedness of $T_n\langle\ s_1,\ldots,s_k;\ t_1,\ldots,t_l\rangle$ means precisely connectedness of $T_n\langle s_1,\ldots,s_k;\ t_1,\ldots,t_l\rangle$ (with duplicates dropped) i.e., $T_n\langle s_1,\ldots,s_k,\ t_1,\ldots,t_l\rangle$.

Remark that $T_n\langle s_1,\ldots,s_i;t_1,\ldots,t_j\rangle$ and $T_n\langle t_1,\ldots,t_j;s_1,\ldots,s_i\rangle$ are obtained from each other by reversing the orientation of all arcs.

Properties of Toeplitz graphs, such as bipartiteness, planarity and colourability, have been studied in [5] and [6]. Circulant graphs, which are special Toeplitz graphs, have been intensively studied (see [1], [2], [4], [8], [12], [13] and [7]). Hamiltonian properties of undirected Toeplitz graphs have been investigated in [3] and [9] and those of directed Toeplitz graphs have been investigated in [11] and [10].

In order to understand the hamiltonian properties of Toeplitz graphs it is important to study the case of small values for k and l, starting of course with small values of s_k, t_l. The hamiltonicity of larger Toeplitz graphs then follows.

Paper [11] investigates the hamiltonicity of the Toeplitz graphs with $s_2 = 2$, and in particular those with $s_3 = 3$. Paper [10] extends this investigation to the case $s_1 = t_1 = 1$ with $s_2 = 3$. Following is a list of main results of these two papers ([11] and [10]).

1. For even t, $T_n\langle 1,2;t\rangle$ is hamiltonian if and only if n is odd.
2. $T_n\langle 1,2;3\rangle$ is hamiltonian if and only if $n = 5$ or $n \equiv 1\,(mod\,3)$.
3. $T_n\langle 1,2;5\rangle$ is hamiltonian for all $n \geqslant 29$.
4. Let $t \geq 7$ be an odd integer. Then $T_n\langle 1,2;t\rangle$ is hamiltonian for all $n > 3t + 5$.
5. $T_n\langle 1,2,3;2\rangle$ is hamiltonian if and only if $n = 4$ or $n \equiv 1\,(mod\,2)$.
6. $T_n\langle 1,2,3;t\rangle$ is hamiltonian for all $t \geq 3$ and n.
7. $T_n\langle s;t\rangle$ is a circuit if and only if $gcd\,(s,\,t) = 1$ and $s + t = n$.
8. For $t \in \{2,4,6\}$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian for all n.
9. $T_n\langle 1,3;1,t\rangle$, where $t(\geq 8)$ is even, is hamiltonian if $n \equiv 0,\ 2,\ 4,\ 6,\ 5,\ 7,\ 9,\ \ldots,\ t-3\ \ (mod\ (t-1))$.
10. $T_n\langle 1,3;1,t\rangle$, where $t(\geq 3)$ is odd, is hamiltonian if and only if n is even.

In this paper we extend the investigation to the case $s_2 = 3$, $s_3 = 4$, still keeping $s_1 = 1$ and $l = 1$. Thus, the Toeplitz graphs treated here have the form $T_n\langle 1, 3, 4; t\rangle$. The main results are: For $t = 2$, $T_n\langle 1, 3, 4; t\rangle$ is hamiltonian for infinitely many n's. For $4 \leq t \leq 9$, it is hamiltonian for all but finitely many n's. For $t \geq 10$, $T_n\langle 1, 3, 4; t\rangle$ is hamiltonian for all n. So in this paper we discuss all the cases for which $T_n\langle 1, 3, 4; t\rangle$ is hamiltonian, and some cases for which $T_n\langle 1, 3, 4; t\rangle$ is non hamiltonian (i.e., for $t = 5$, 8 and two cases for $t = 4$) and leave the remaining ones as conjectures.

We underline a pair of consecutive vertices (say $n-1$ and n) as $\underline{n-1,\ n}$ to emphasize that $(n-1,\ n)$ is an arc in the hamiltonian circuit.

2 Toeplitz graphs $T_n\langle 1, 3, 4; t\rangle$ with $t < 4$

We start with the following lemma which is included in Theorem 1 of [10].

Lemma 1. *If* $\gcd(s, t) = 1$ *then* $T_{s+t}\langle s; t\rangle$ *is a circuit.*

Theorem 1. $T_n\langle 1, 3, 4; 2\rangle$ *is hamiltonian for* $n \in \{5, 7\}$ *and all* $n \cong 0, 3$ *or* 4 *modulo* 6.

Proof. $T_5\langle 1, 3, 4; 2\rangle$ is hamiltonian by Lemma 1.

Fig. 1 shows why $T_7\langle 1, 3, 4; 2\rangle$ is hamiltonian.

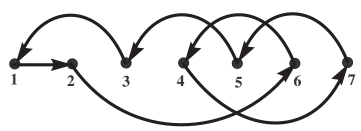

Fig. 1.

Fig. 2 shows the hamiltonian circuit (1, 2, 6, $\underline{4,\ 5}$, 3, 1) in $T_6\langle 1, 3, 4; 2\rangle$, which contains the arc (4, 5).

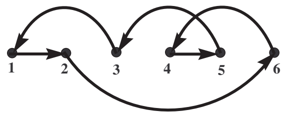

Fig. 2.

Fig. 3 shows the hamiltonian circuit (1, 5, 9, 7, 8, 6, 4, 2, 3, 1) in $T_9\langle 1, 3, 4; 2\rangle$, which contains the arc (7, 8).

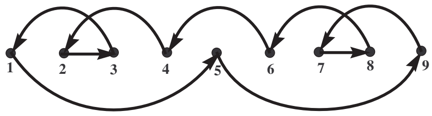

Fig. 3.

Fig. 4 shows the hamiltonian circuit (1, 4, 2, 6, 10, 8, 9, 7, 5, 3, 1) in $T_{10}\langle 1, 3, 4; 2\rangle$, which contains the arc (8, 9).

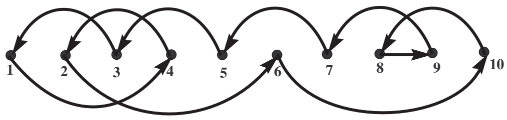

Fig. 4.

Since any hamiltonian circuit of $T_n\langle 1, 3, 4; 2\rangle$ which contains the arc $(n-2,\ n-1)$ can be transformed into a hamiltonian circuit of $T_{n+6}\langle 1, 3, 4; 2\rangle$ containing the arc $(n+4,\ n+5)$ by replacing the arc $(n-2,\ n-1)$ with the path $(n-2,\ n+2,\ n+6,\ \underline{n+4,\ n+5},\ n+3,\ n+1,\ n-1)$, the theorem follows. □

Theorem 2. $T_n\langle 1,3,4;3\rangle$ *is hamiltonian for* $n \in \{5,\ 6,\ 7,\ 9\}$.

Proof. Fig. 5 shows the hamiltonian circuit (1, 5, 2, 3, 4, 1) in $T_5\langle 1,3,4;3\rangle$.

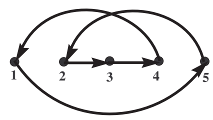

Fig. 5.

Fig. 6 shows the hamiltonian circuit (1, 2, 5, 6, 3, 4, 1) in $T_6\langle 1,3,4;3\rangle$.

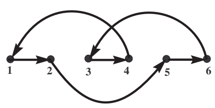

Fig. 6.

$T_7\langle 1,3,4;3\rangle$ is hamiltonian by Lemma 1. Fig. 7 shows a hamiltonian circuit (1, 2, 5, 8, 9, 6, 3, 7, 4, 1) in $T_9\langle 1,3,4;3\rangle$.

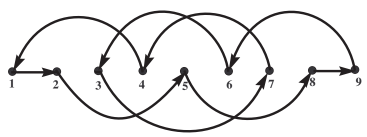

Fig. 7.

This finishes the proof. □

We ignore whether $T_n\langle 1,3,4;3\rangle$ is hamiltonian for any $n \notin \{5,\ 6,\ 7,\ 9\}$.

3 Toeplitz graphs $T_n\langle 1, 3, 4; t\rangle$ with $4 \leq t \leq 9$

Theorem 3. *$T_n\langle 1, 3, 4; 4\rangle$ is hamiltonian for $n \in \{5, 7, 8, 9, 11, 14, 15, 17, 18, 20, 21\}$ and all $n \geq 23$.*

Proof. $T_5\langle 1, 3, 4; 4\rangle$ and $T_7\langle 1, 3, 4; 4\rangle$ are hamiltonian by Lemma 1. The first includes the circuit $T_5\langle 1; 4\rangle$, which contains the arc (3, 4). Fig. 8 shows why $T_9\langle 1, 3, 4; 4\rangle$ is hamiltonian.

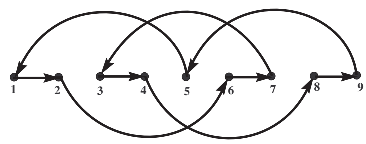

Fig. 8.

Fig. 9 shows the hamiltonian circuit (1, 2, 3, 7, 10, 6, 9, 13, 14, 15, 11, 12, 8, 4, 5, 1) in $T_{15}\langle 1, 3, 4; 4\rangle$.

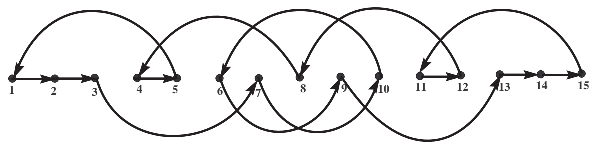

Fig. 9.

In Fig. 10 we see the hamiltonian circuit (1, 2, 3, 7, 10, 6, 9, 13, 17, 20, 16, 19, 23, 24, 25, 21, 22, 18, 14, 15, 11, 12, 8, 4, 5, 1) in $T_{25}\langle 1, 3, 4; 4\rangle$.

Any hamiltonian circuit of $T_n\langle 1, 3, 4; 4\rangle$ which contains the arc $(n-2, n-1)$ can be transformed into a hamiltonian circuit of $T_{n+3}\langle 1, 3, 4; 4\rangle$ containing the arc $(n+1, n+2)$ by replacing the arc $(n-2, n-1)$ with the path $(n-2, n+1, n+2, n+3, n-1)$, and this finishes the proof. □

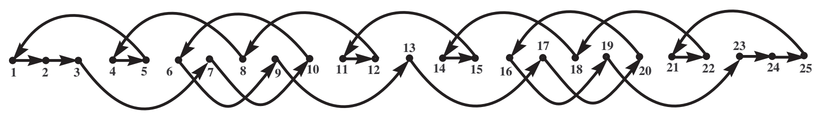

Fig. 10.

Theorem 4. $T_6\langle 1,3,4;4\rangle$ *is non-hamiltonian.*

Proof. Suppose on contrary that $T_6\langle 1,3,4;4\rangle$ is hamiltonian and $H = H_{1\to 6} \cup H_{6\to 1}$ is a hamiltonian circuit in $T_6\langle 1,3,4;4\rangle$. Then for every vertex v in H, we have $d^-(v) = 1 = d^+(v)$.

$T_6\langle 1,3,4;4\rangle$ has only two decreasing arcs namely $(6,2)$ and $(5,1)$ and both of them are in $H_{6\to 1}$ because $d^-(1) = 1 = d^+(6)$. So $H_{6\to 1}$ would be $H_{6\to 1} = (6,2) \cup (2,5) \cup (5,1)$. Now $H_{1\to 6}$ must contain the arc $(1,4)$ but then $H_{1\to 6}$ would be stuck at vertex 4 and also the vertex 3 would be lost. So a contradiction. □

Definition

The vertices $V = \{u_1, u_2, \ldots, u_k\}$ are said to be ***consecutive vertices*** of order $k \geq 2$ if there exists an arc of length one between u_1 and u_2, between u_2 and u_3, so on, and between u_{k-1} and u_k. Two set of consecutive vertices say V_1 and V_2 are ***disjoint*** if there does not exist an arc of legth one between any vertex of V_1 and any vertex of V_2.

Theorem 5. $T_{10}\langle 1,3,4;4\rangle$ *is non-hamiltonian.*

Proof. Suppose on contrary that $T_{10}\langle 1,3,4;4\rangle$ is hamiltonian and $H = H_{1\to 10} \cup H_{10\to 1}$ is a hamiltonian circuit in $T_{10}\langle 1,3,4;4\rangle$. Then for every vertex v in H, we have $d^-(v) = 1 = d^+(v)$.
Let

$$V(H_{10\to 1} \setminus \{1,10\}) = V_1 \cup V_2 \cdots \cup V_k$$

where each V_i is a disjoint set of consecutive vertices of order ≥ 2. But then order of each V_i should be not more than 3 because $H_{1\to 10}$ has no arc of length more than 3. Thus $|V_i| = 2$ or 3.

Let A be the set of all decreasing arcs in $T_{10}\langle 1,3,4;8\rangle$, i.e., $A = \{(10,6), (9,5), (8,4), (7,3), (6,2), (5,1)\}$. The arcs $(10,6)$ and $(5,1)$ both are in $H_{10\to 1}$ because $d^-(1) = 1 = d^+(10)$ in H. But $H_{10\to 1}$ cannot have only these two arcs as its decreasing arcs because otherwise $H_{10\to 1}$ would be stuck at vertex 6. Let B be the set of all decreasing arcs in $H_{10\to 1}$. Thus $3 \le |B| \le 6$. Four cases arise as per number of decreasing arcs in $H_{10\to 1}$.

Case 1. If $|B| = 6$.

Thus $B = A$ and $V(H_{10\to 1} \setminus \{1,10\}) = \{2,3,4,5,6,7,8,9\} = V_1$, where V_1 is a set of consecutive vertices, but $|V_1| > 3$ so a contradiction.

Case 2. If $|B| = 5$. Since $(10,6),(5,1) \in B$, four subcases arise.

1. $(9,5)$, $(8,4)$, $(7,3) \in B$.
   Since $V(H_{10\to 1} \setminus \{1,10\}) = \{3,4,5,6,7,8,9\} = V_1$ but $|V_1| > 3$ so a contradiction.
2. $(9,5)$, $(8,4)$, $(6,2) \in B$.
   Thus $H_{10\to 1} = (10,6) \cup (6,2) \cup P_{2\to 8} \cup (8,4) \cup P_{4\to 9} \cup (9,5) \cup (5,1)$ but then $H_{10\to 1}$ would be stuck at $P_{2\to 8}$ because the only possibilty for the path $P_{2\to 8}$ in $H_{10\to 1}$ is $P_{2\to 8} = (2,3) \cup P_{3\to 8}$ but in this case $H_{10\to 1} \setminus \{1,10\}$ would have more than 3 consecutive vertices.
3. $(9,5)$, $(7,3)$, $(6,2) \in B$.
   $H_{10\to 1} = (10,6) \cup (6,2) \cup P_{2\to 7} \cup (7,3) \cup P_{3\to 9} \cup (9,5) \cup (5,1)$ but then $H_{10\to 1}$ would be stuck at $P_{2\to 7}$ because $P_{2\to 7}$ cannot go beyond vertex 2.
4. $(8,4)$, $(7,3)$, $(6,2) \in B$.
   $V(H_{10\to 1} \setminus \{1,10\}) = \{2,3,4,5,6,7,8\} = V_1$ but $|V_1| > 3$ so a contradiction.

Case 3. If $|B| = 4$. Since $(10,6),(5,1) \in B$, six subcases arise.

1. $(9,5)$, $(7,3) \in B$.
   Thus $H_{10\to 1} = (10,6) \cup (6,7) \cup (7,3) \cup P_{3\to 9} \cup (9,5) \cup (5,1)$ but then $H_{10\to 1}$ would be stuck at $P_{3\to 9}$ because the only possibilty for the path $P_{3\to 9}$ in $H_{10\to 1}$ is $P_{3\to 9} = (3,4) \cup P_{4\to 9}$ but in this case $H_{10\to 1} \setminus \{1,10\}$ would have more than 3 consecutive vertices.

2. $(9, 5)$, $(6, 2) \in B$.
   Thus $H_{10\to 1} = (10, 6) \cup (6, 2) \cup P_{2\to 9} \cup (9, 5) \cup (5, 1)$ but then $H_{10\to 1}$ would be stuck at $P_{2\to 9}$ because the only possibilty for the path $P_{2\to 9}$ in $H_{10\to 1}$ is $P_{2\to 9} = (2, 3) \cup (3, 7) \cup P_{7\to 9}$ which would stuck at vertex $\cup P_{7\to 9}$ as cannot go beyong vertex 7.
3. $(8, 4)$, $(7, 3) \in B$.
   $V(H_{10\to 1} \setminus \{1, 10\}) = \{3, 4, 5, 6, 7, 8\} = V_1$ but $|V_1| > 3$ so a contradiction.
4. $(8, 4)$, $(6, 2) \in B$.
   Thus $H_{10\to 1} = (10, 6) \cup (6, 2) \cup P_{2\to 8} \cup (8, 4) \cup (4, 5) \cup (5, 1)$ but then $H_{10\to 1}$ would be stuck at $P_{2\to 8}$ because the only possibilty for the path $P_{2\to 8}$ in $H_{10\to 1}$ is $P_{2\to 8} = (2, 3) \cup P_{3\to 8}$ but in this case $H_{10\to 1}$ would have more than 3 consecutive vertices.
5. $(7, 3)$, $(6, 2) \in B$.
   $H_{10\to 1} = (10, 6) \cup (6, 2) \cup P_{2\to 7} \cup (7, 3) \cup (3, 5) \cup (5, 1)$ but then $H_{10\to 1}$ would be stuck at $P_{2\to 7}$ as cannot go beyond vertex 2.
6. $(9, 5)$, $(8, 4) \in B$.
   Thus $H_{10\to 1} = (10, 6) \cup P_{6\to 8} \cup (8, 4) \cup P_{8\to 4} \cup P_{4\to 9} \cup (9, 5) \cup (5, 1)$ but then $H_{10\to 1}$ would be stuck at $P_{6\to 8}$.

Case 4. If $|B| = 4$. Since $(10, 6), (5, 1) \in B$, four subcases arise.

1. $(9, 5) \in B$.
   Thus $H_{10\to 1} = (10, 6) \cup P_{6\to 9} \cup (9, 5) \cup (5, 1)$ but then $H_{10\to 1}$ would be stuck at $P_{6\to 9}$ because the only possibilty for the path $P_{6\to 9}$ in $H_{10\to 1}$ is $P_{6\to 9} = (6, 9)$ but in this case $H_{1\to 10}$ would be $H_{1\to 10} = (1, 2) \cup (2, 3) \cup (3, 4) \cup (4, 7) \cup (7, 8) \cup P_{8\to 10}$ which would be stuck at vertex 8 in $P_{8\to 10}$.
2. $(8, 4) \in B$.
   Thus $H_{10\to 1} = (10, 6) \cup P_{6\to 8} \cup (8, 4) \cup (4, 5) \cup (5, 1)$ but then $H_{10\to 1}$ would be stuck at $P_{6\to 8}$.
3. $(7, 3) \in B$.
   $H_{10\to 1} = (10, 6) \cup (6, 7) \cup (7, 3) \cup P_{3\to 5} \cup (5, 1)$ but then $H_{10\to 1}$ would be stuck at $P_{3\to 5}$.
4. $(6, 2) \in B$.
   Thus $H_{10\to 1} = (10, 6) \cup (6, 2) \cup (2, 5) \cup (5, 1)$ but then $H_{1\to 10}$ would be stuck at vertex 1.

   Thus In each case there is a contradiction. Hence $T_{10}\langle 1, 3, 4; 4\rangle$ is non-hamiltonian.

□

Theorem 6. $T_n\langle 1,3,4;5\rangle$ *is hamiltonian for all n if and only if* $n \neq 7$.

Proof. Claim 1. For $n \in \{8, 11\}$, $T_n\langle 1,3,4;5\rangle$ is hamiltonian.

Indeed $T_8\langle 1,3,4;5\rangle$ has a hamiltonian circuit (1, 4, 7, 2, 5, 8 , 3, 6, 1), and $T_{11}\langle 1,3,4;5\rangle$ has a hamiltonian circuit (1, 2, 5, 9, 4, 8, 3, 7 , 10 , 11 , 6 , 1) (see Figs. 11-12).

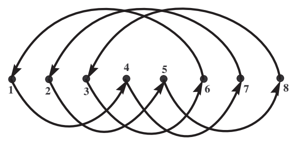

Fig. 11.

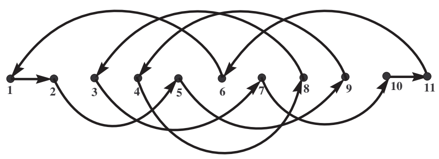

Fig. 12.

Claim 2. For $n \in \{6, 9, 12, 15\}$, $T_n\langle 1,3,4;5\rangle$ has a hamiltonian circuit containing the arc $(n-2,\ n-1)$.

Indeed $T_6\langle 1,3,4;5\rangle$ contains the hamiltonian circuit (1, 2, 3, $\underline{4,\ 5}$, 6). In $T_9\langle 1,3,4;5\rangle$ the circuit (1, 2, 3, $\underline{7,\ 8}$, 9, 4, 5, 6, 1) is hamiltonian, in $T_{12}\langle 1,3,4;5\rangle$ the circuit (1, 4, 5, 8, 9, $\underline{10,\ 11}$, 12, 7, 2, 3, 6, 1) is hamiltonian, and in $T_{15}\langle 1,3,4;5\rangle$ the circuit (1, 4, 5, 8, 9, $\underline{13,\ 14}$, 15, 10, 11, 12, 7, 2, 3, 6, 1), is hamiltonian (see Figs. 13-16).

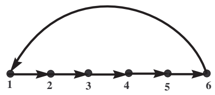

Fig. 13.

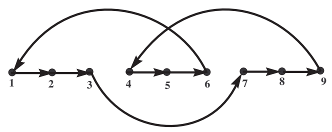

Fig. 14.

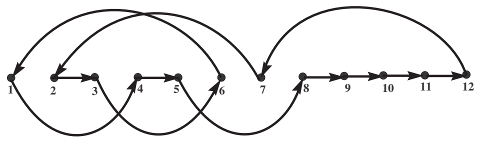

Fig. 15.

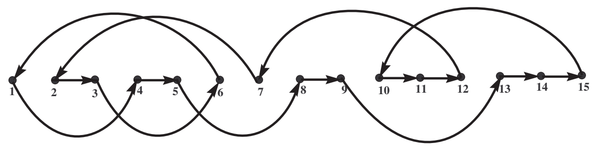

Fig. 16.

Starting from the above values of $n \in \{6,\, 9,\, 12,\, 15\}$, we can extend a hamiltonian circuit in $T_n\langle 1,3,4;5\rangle$ containing the arc $(n-2,\, n-1)$ to a hamiltonian circuit in $T_{n+4}\langle 1,3,4;5\rangle$ with the same property by replacing the arc $(n-2,\, n-1)$ with the path

$$(n-2,\, n+1,\, \underline{n+2,\, n+3},\, n+4,\, n-1).$$

Since 6, 9, 12, 15 are representatives in each of the various rest classes modulo 4, it follows that $T_n\langle 1,3,4;5\rangle$ is hamiltonian for $n = 6$, $n = 9$, $n = 10$ and $n \geqslant 12$. This together with Claim 1 shows that if $n \neq 7$ then $T_n\langle 1,3,4;5\rangle$ is hamiltonian for all n.

Conversely, Let $n = 7$. Suppose on contrary that $T_7\langle 1,3,4;5\rangle$ is hamiltonian and $H = H_{1\to 7} \cup H_{7\to 1}$ is a hamiltonian circuit in $T_7\langle 1,3,4;5\rangle$. Clearly, $H_{7\to 1}$ contains both arcs $(6,1)$ and $(7,2)$, otherwise vertex 1, respectively vertex 7, would be lost. Therefore, the subpath of H from vertex 1 to vertex 7 must be $(1,4,7)$. Now, the only paths from vertex 2 to vertex 6 are $(2,6)$, $(2,3,6)$ and $(2,5,6)$, and each time at least one point remains unvisited which contradicts our assumption. □

Theorem 7. *$T_n\langle 1,3,4;6\rangle$ is hamiltonian for all n.*

Proof. Claim 1. $T_9\langle 1,3,4;6\rangle$ is hamiltonian.

Indeed a hamiltonian circuit in $T_9\langle 1,3,4;6\rangle$ is (1, 4, 8, 2, 5, 6, 9, 3, 7, 1), see Fig. 17.

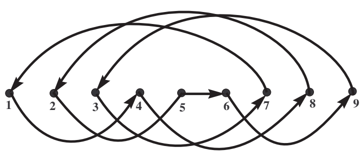

Fig. 17.

Claim 2. For $n \in \{7, 8, 10, 11, 14\}$, $T_n\langle 1,3,4;6\rangle$ has a hamiltonian circuit containing the arc $(n-2,\ n-1)$.

Indeed $T_7\langle 1,3,4;6\rangle$ has the hamiltonian circuit $T_7\langle 1;6\rangle$. In $T_8\langle\ 1,3,4;6\rangle$ we find the hamiltonian circuit (1, 4, 5, 8, 2, 3, 6, 7, 1), in $T_{10}\langle 1,3,4;6\rangle$ the circuit (1, 2, 5, 8, 9, 3, 6, 10, 4, 7, 1), in $T_{11}\langle 1,3,4;6\rangle$ the circuit (1, 2, 3, 4, 8, 9, 10, 11, 5, 6, 7, 1), and in $T_{14}\langle 1,3,4;6\rangle$ the circuit (1, 4, 5, 6, 9, 10, 11, 12, 13, 14, 8, 2, 3, 7, 1) (see Figs. 18-21).

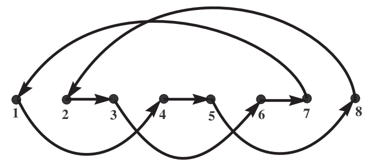

Fig. 18.

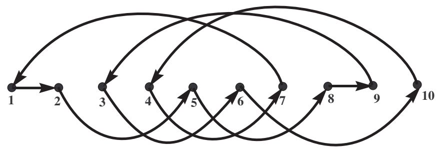

Fig. 19.

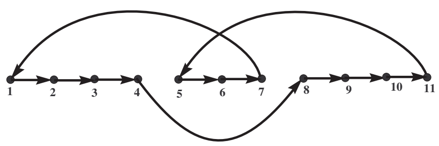

Fig. 20.

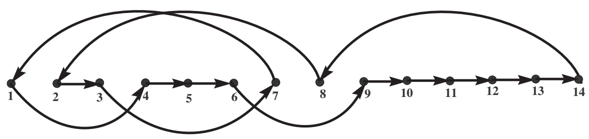

Fig. 21.

Starting from the above values of $n \in \{7,\ 8,\ 10,\ 11,\ 14\}$, we can extend a hamiltonian circuit in $T_n\langle 1,3,4;6\rangle$ containing the arc $(n-2,\ n-1)$ to a hamiltonian circuit in $T_{n+5}\langle 1,3,4;6\rangle$ with the same property by replacing the arc $(n-2,\ n-1)$ with the path

$$(n-2,\ n+1,\ n+2,\ \underline{n+3,\ n+4},\ n+5,\ n-1).$$

Since 7, 8, 10, 11, 14 are representatives in each of the various rest classes modulo 5, it follows that $T_n\langle 1,3,4;6\rangle$ is hamiltonian for all $n \neq 9$. This together with Claim 1 shows that $T_n\langle 1,3,4;6\rangle$ is hamiltonian for all n. □

Theorem 8. *$T_n\langle 1,3,4;7\rangle$ is hamiltonian for all n.*

Proof. Claim 1. $T_{11}\langle 1,3,4;7\rangle$ is hamiltonian.

Indeed in $T_{11}\langle 1,3,4;7\rangle$ the circuit (1, 2, 5, 9, 10, 3, 6, 7, 11, 4, 8, 1) is hamiltonian (see Fig. 22).

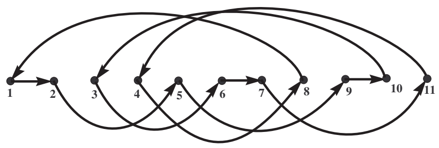

Fig. 22.

Claim 2. For $n \in \{8,\ 9,\ 10,\ 12,\ 16\}$, $T_n\langle 1,3,4;7\rangle$ has a hamiltonian circuit containing the arc $(n-3,\ n-2)$.

Indeed $T_8\langle 1,3,4;7\rangle$ has the hamiltonian circuit $T_8\langle 1;7\rangle$, in $T_9\langle$ 1, 3, 4; 7$\rangle$ the circuit (1, 4, 5, 9, 2, 3, $\underline{6,\ 7}$, 8, 1) is hamiltonian, in $T_{10}\langle 1,3,4;7\rangle$ we find the hamiltonian circuit ($\underline{1,\ 2}$, 5, 6, 9, 10, 3, 4, $\underline{7,\ 8}$, 1), in $T_{12}\langle 1,3,4;7\rangle$ the circuit (1, 2, 6, $\underline{9,\ 10}$, 3, 4, 7, 11, 12 , 5, 8, 1), and in $T_{16}\langle 1,3,4;7\rangle$ the circuit (1, 5, 6, 10, $\underline{11,\ 12}$, $\underline{13,\ 14}$, 15, 16, 9, 2, 3, 4, 7, 8, 1) (see Figs. 23-26).

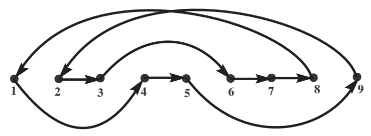

Fig. 23.

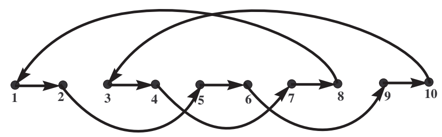

Fig. 24.

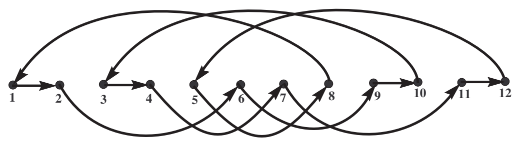

Fig. 25.

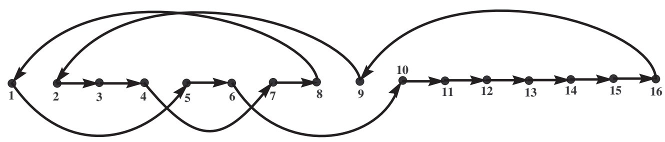

Fig. 26.

From the initial values $n \in \{8, 9, 10, 12, 16\}$, we inductively extend a hamiltonian circuit in $T_n\langle 1, 3, 4; 7\rangle$ containing the arc $(n-3,\, n-2)$ to a hamiltonian circuit in $T_{n+5}\langle 1, 3, 4; 7\rangle$ with the same property, by replacing the arc $(n-3,\, n-2)$ with the path

$$(n-3,\, n+1,\, \underline{n+2,\, n+3},\, n+4,\, n+5,\, n-2).$$

Since 8, 9, 10, 12, 16 are representatives in each of the various rest classes modulo 5, it follows that $T_n\langle 1, 3, 4; 7\rangle$ is hamiltonian for all $n \neq 11$. This together with Claim 1 shows that $T_n\langle 1, 3, 4; 7\rangle$ is hamiltonian for all n. □

Theorem 9. $T_n\langle 1, 3, 4; 8\rangle$ *is hamiltonian for all n different from* 14.

Proof. Claim 1. $T_{12}\langle 1, 3, 4; 8\rangle$ is hamiltonian.

Indeed $T_{12}\langle 1, 3, 4; 8\rangle$ has a hamiltonian circuit (1, 2, 3, 6, 7, 10, 11, 12, 4, 5, 8, 9, 1), see Fig. 27.

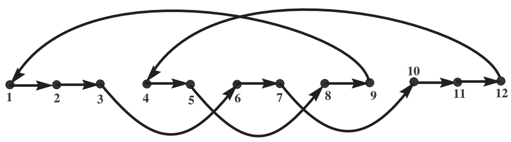

Fig. 27.

Claim 2. For $n \in \{9, 10, 11, 13, 18, 20\}$, $T_n\langle 1, 3, 4; 8\rangle$ has a hamiltonian circuit containing the arc $(n-3,\, n-2)$.

Indeed $T_9\langle 1, 3, 4; 8\rangle$ has the hamiltonian circuit $T_9\langle 1; 8\rangle = (1, 2, 3, 4, 5, \underline{6, 7}, 8, 9, 1)$. In $T_{10}\langle 1, 3, 4; 8\rangle$ the circuit $(1, 5, 6, 10, 2, 3, 4, \underline{7, 8}, 9, 1)$ is hamiltonian, see Fig. 28.

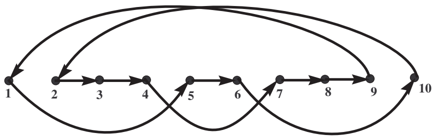

Fig. 28.

In $T_{11}\langle 1, 3, 4; 8\rangle$ the circuit (1, 2, 5, 6, 7, 10, 11, 3, 4, 8, 9, 1) is hamiltonian, see Fig. 29.

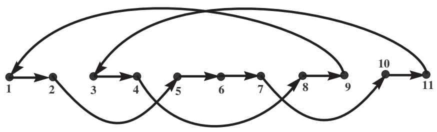

Fig. 29.

$T_{13}\langle 1, 3, 4; 8\rangle$ is spanned by the circuit (1, 2, 6, 10, 11, 3, 4, 7, 8, 12, 13, 5, 9, 1), $T_{18}\langle 1, 3, 4; 8\rangle$ by (1, 5, 6, 7, 11, 12, 13, 14, 15, 16, 17, 18, 10, 2, 3, 4, 8, 9, 1), and $T_{20}\langle 1, 3, 4; 8\rangle$ by (1, 5, 6, 7, 11, 14, 15, 16, 19, 20, 12, 13, 17, 18, 10, 2, 3, 4, 8, 9, 1) (see Figs. 30-32).

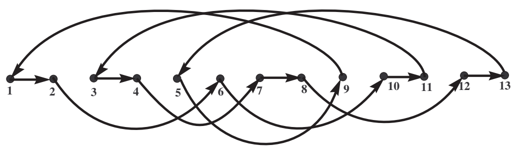

Fig. 30.

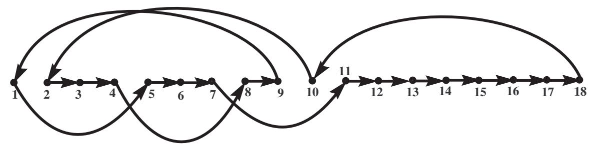

Fig. 31.

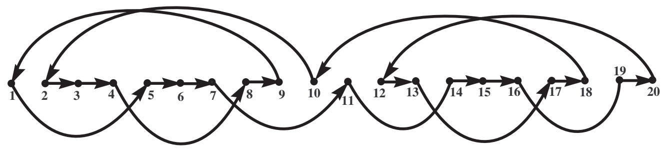

Fig. 32.

Starting from the above values of $n \in \{9,\ 10,\ 11,\ 13,\ 18,\ 20\}$, we can extend a hamiltonian circuit of $T_n\langle 1,3,4;8\rangle$ containing the arc $(n-3,\ n-2)$ to a hamiltonian circuit in $T_{n+6}\langle 1,3,4;8\rangle$ with the same property by replacing the arc $(n-3,\ n-2)$ with the path $(n-3,\ n+1,\ n+2,\ \underline{n+3,\ n+4},\ n+5,\ n+6,\ n-2)$.

Since 9, 10, 11, 13, 18, 20 are representatives in each of the various rest classes modulo 6, it follows that $T_n\langle 1,3,4;8\rangle$ is hamiltonian for $n =$ 9, 10, 11, 13 and for all $n \geqslant 15$. This together with Claim 1 shows that $T_n\langle 1,3,4;8\rangle$ is hamiltonian for all $n \neq 14$. □

Theorem 10. $T_{14}\langle 1,3,4;8\rangle$ *is non-hamiltonian.*

Proof. Suppose on contrary that $T_{14}\langle 1,3,4;8\rangle$ is hamiltonian and $H = H_{1\to 14} \cup H_{14\to 1}$ is a hamiltonian circuit in $T_{14}\langle 1,3,4;8\rangle$. Then for every vertex v in H, we have $d^-(v) = 1 = d^+(v)$. The vertices which are not covered by $H_{14\to 1}$ would be covered by $H_{1\to 14}$, and since increasing arcs in $H_{1\to 14}$ are of length 1, 3 and 4 only, so clearly $H_{14\to 1}$ would not use more than three consecutive vertices.

Let A be the set of all decreasing arcs in $T_{14}\langle 1,3,4;8\rangle$, i.e., $A = \{(14,6),$ $(13,5)$, $(12,4)$, $(11,3)$, $(10,2)$, $(9,1)\}$, and B be the set of all decreasing arcs in H, (clearly $B \subseteq A$). Since $d^-(1) = d^+(14) = 1$ in $T_{14}\langle 1,3,4;8\rangle$, so $(14,6)$, $(9,1) \in B$. Clearly, $(14,6),(9,1) \in E(H_{14\to 1})$. H cannot have only these two arcs as its decreasing arcs, because otherwise $(6,9) \in E(H_{14\to 1})$ but in that case $H_{1\to 14}$ cannot cover all of the remaining vertices $\{2, 3, 4, 5, 7, 8, 10, 11, 12, 13\}$ of H (as $H_{1\to 14}$ would be stuck at vertex 5), Thus $|B| > 2$. Four cases arise as per number of decreasing arcs in H (other than $(14,6)$ and $(9,1)$), i.e., $|B \setminus \{(14,6),(9,1)\}|$.

Case 1. If $|B \setminus \{(14,6),(9,1)\}| = 4$, then $(10,2)$, $(11,3)$, $(12,4)$, $(13,5) \in B$, which implies $d^-(2) = d^-(3) = d^-(4) = d^-(5) = 1$ in H. But then $H_{1\to 14}$ cannot go beyond vertex 1, so it would be stuck at vertex 1.

Case 2. If $|B \setminus \{(14,6),(9,1)\}| = 3$, then four subcases arise.

1. $(10,2)$, $(11,3)$, $(12,4) \in B$.
   Then clearly $(1,5) \in A(H)$, but then H would be stuck at vertex 2.
2. $(10,2)$, $(11,3)$, $(13,5) \in B$.
   Then Cleary, $(1,4) \in A(H)$, but then H would be stuck at vertex 2.
3. $(10,2)$, $(12,4)$, $(13,5) \in B$.
   Then H cannot go beyond vertex 1, so it would be stuck at vertex 1.
4. $(11,3)$, $(12,4)$, $(13,5) \in B$.
   Then Cleary, $(1,2) \in A(H)$, but then H would be stuck at vertex 2.

Case 3. If $|B \setminus \{(14,6),(9,1)\}| = 2$, then six subcases arise.

1. $(10,2)$, $(11,3) \in B$.
   Clearly, $(2,5) \in A(H)$, which implies $(1,4) \in A(H) \Rightarrow (3,7) \in A(H) \Rightarrow (4,8) \in A(H) \Rightarrow (5,9) \in A(H) \Rightarrow (6,10) \in A(H)$. But then H would be stuck at vertex 7 (otherwise we would have a shorter circuit).
2. $(10,2)$, $(12,4) \in B$.
   Clearly, $(2,3,7)$ and $(1,5,8)$ is a path in H. But then H would be stuck at vertex 4.
3. $(10,2)$, $(13,5) \in B$.
   Clearly, $(1,4) \in A(H)$, which implies $(2,3,7)$ is a path in $H \Rightarrow (4,8) \in A(H) \Rightarrow (5,9) \in A(H) \Rightarrow (6,10) \in A(H) \Rightarrow (7,11) \in A(H) \Rightarrow (8,12) \in A(H)$. But then H would be stuck at vertex 12 (otherwise we would have a shorter circuit).
4. $(11,3)$, $(12,4) \in B$.
   Clearly, $(1,2,5,8)$ is a path in $H_{1\to 14}$ and $(3,7) \in A(H)$. But then H would be stuck at vertex 4.
5. $(11,3)$, $(13,5) \in B$.
   Clearly, $(1,2) \in E(H_{1\to 14})$, but then $H_{1\to 14}$ would be stuck at vertex 2.

6. $(12, 4)$, $(13, 5) \in B$.
Clearly, $(1, 2, 3, 7)$ is a path in $H_{1\to 14}$, which implies $(4, 8) \in A(H) \Rightarrow (5, 9) \in A(H) \Rightarrow (6, 10) \in A(H) \Rightarrow (7, 11) \in A(H)$, but then H would be stuck at vertex 8 (otherwise we would have a shorter circuit).

Case 4. If $|B \setminus \{(14, 6), (9, 1)\}| = 1$, then four subcases arise.

1. $(10, 2) \in B$.
Since $(14, 6)$, $(9, 1) \in E(H_{14\to 1})$, so clearly, $(6, 9) \notin E(H_{14\to 1})$, (otherwise $H_{1\to 14} = P_{1\to 10} \cup (10, 2) \cup P_{2\to 14}$, where $P_{1\to 10}$ should be $(1, 4, 7, 10)$, but then $P_{2\to 14}$ would be stuck at vertex 3). Since $(6, 9) \notin E(H_{14\to 1})$, so $(10, 2) \in E(H_{14\to 1})$, which implies $(2, 3) \in E(H_{14\to 1})$. Clearly, $H_{14\to 1} = (14, 6) \cup P_{6\to 10} \cup (10, 2) \cup P_{2\to 9} \cup (9, 1)$, where $P_{6\to 10} = (6, 7, 10)$ or $(6, 10)$. For $P_{6\to 10} = (6, 7, 10)$, we cannot find any path $P_{2\to 9}$ such that $H_{14\to 1}$ would use at most three consecutive vertices. Thus $P_{6\to 10} = (6, 10)$, and in this case $P_{2\to 9}$ would be $(2, 3, 4, 8, 9)$. But then $H_{1\to 14}$ would be stuck at vertex 5.
2. $(11, 3) \in B$.
Since $(14, 6)$, $(9, 1) \in E(H_{14\to 1})$, so clearly, $(6, 9) \notin E(H_{14\to 1})$ (otherwise $H_{1\to 14} = P_{1\to 11} \cup (11, 3) \cup P_{3\to 14}$, where $P_{1\to 11}$ should be $(1, 2, 5, 8, 11)$, but then $P_{3\to 14}$ would be stuck at vertex 10, because otherwise some vertices would be lost). Since $(6, 9) \notin E(H_{14\to 1})$, so $(11, 3) \in E(H_{14\to 1})$, which implies $H_{1\to 14} = (1, 2, 5, 8, 12, 13, 14)$. But since $H_{14\to 1} = (14, 6) \cup P_{6\to 11} \cup (11, 3) \cup P_{3\to 9} \cup (9, 1)$, we cannot find a path $P_{3\to 9}$ as it would be stuck at vertex 7.
3. $(12, 4) \in B$.
Since $(14, 6)$, $(9, 1) \in E(H_{14\to 1})$, so clearly, $(6, 9) \notin E(H_{14\to 1})$ (otherwise $H_{1\to 14} = P_{1\to 12} \cup (12, 4) \cup P_{4\to 14}$, where $(4, 5, 8, 11)$ should be a path in $P_{4\to 14}$, but this path would be stuck at vertex 11 because otherwise vertex 13 would be lost). Since $(6, 9) \notin E(H_{14\to 1})$, so $(12, 4) \in E(H_{14\to 1})$. Clearly, $H_{14\to 1} = (14, 6) \cup P_{6\to 12} \cup (12, 4) \cup P_{4\to 9} \cup (9, 1)$, then $P_{6\to 12}$ must be $(6, 7, 11, 12)$ (otherwise $H_{14\to 1}$ uses more than three consecutive vertices), but then we cannot find any path $P_{4\to 9}$ such that $H_{14\to 1}$ would use at most three consecutive vertices.
4. $(13, 5) \in B$.
Since $(14, 6)$, $(9, 1) \in E(H_{14\to 1})$, so clearly, $(6, 9) \notin E(H_{14\to 1})$ (otherwise $H_{1\to 14} = P_{1\to 13} \cup (13, 5) \cup P_{5\to 14}$, where $P_{5\to 14}$ should be $(5, 8, 11, 14)$, but then $P_{1\to 13}$ would be stuck at vertex 10, because otherwise vertex 12 would be lost). Since $(6, 9) \notin E(H_{14\to 1})$, so $(13, 5) \in E(H_{14\to 1})$. Clearly, $H_{14\to 1} = (14, 6) \cup P_{6\to 13} \cup (13, 5) \cup P_{5\to 9} \cup (9, 1)$. Here $P_{6\to 13} = (6, 10, 13)$ or $(6, 7, 10, 13)$ and $P_{5\to 9} = (5, 8, 9)$ or $(5, 9)$. If $P_{6\to 13} = (6, 10, 13)$ and $P_{5\to 9} = (5, 8, 9)$, then $(1, 2, 3, 4,7,11,12)$ would be a path in $H_{1\to 14}$. If $P_{6\to 13} = (6, 10, 13)$ and $P_{5\to 9} = (5, 9)$, then $(1, 2, 3, 4,7,8,11,12)$ would be a path in $H_{1\to 14}$. If $P_{6\to 13} = (6, 7, 10, 13)$ then $P_{5\to 9}$ must be $(5, 9)$

(otherwise $H_{14\to 1}$ would use more than three consecutive vertices), then $(1, 2, 3, 4, 8, 11, 12)$ would be a path in $H_{1\to 14}$. But in each case $H_{1\to 14}$ would be stuck at vertex 12.

A contradiction occurs in each case, hence $T_{14}\langle 1, 3, 4; 8\rangle$ is non-hamiltonian. This finishes the proof □

Theorem 11. *$T_n\langle 1, 3, 4; 9\rangle$ is hamiltonian for all n different from* 15.

Proof. Claim 1. $T_{12}\langle 1, 3, 4; 9\rangle$ is hamiltonian.

Indeed, a hamiltonian circuit in $T_{12}\langle 1, 3, 4; 9\rangle$ is (1, 2, 6, 7, 11, 12, 3, 4, 5, 8, 9, 10, 1), see Fig. 33.

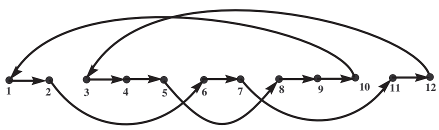

Fig. 33.

Claim 2. For $n \in$ {10, 11, 13, 14, 16, 17, 20, 23}, $T_n\langle 1, 3, 4; 9\rangle$ has a hamiltonian circuit containing the arc ($n - 2$, $n - 1$).

Indeed $T_{10}\langle 1, 3, 4; 9\rangle$ has the hamiltonian circuit $T_{10}\langle 1; 9\rangle$. In $T_{11}\langle$ 1, 3, 4; 9$\rangle$ the circuit (1, 5, 6, 7, 11, 2, 3, 4, 8, 9, 10, 1) is hamiltonian, in $T_{13}\langle$1, 3, 4; 9$\rangle$ the circuit (1, 2, 3, 6, 7, 8, 11, 12, 13, 4, 5, 9, 10, 1) is hamiltonian, in $T_{14}\langle$1, 3, 4; 9$\rangle$ we find the hamiltonian circuit (1, 2, 3, 4, 7, 8, 11, 12, 13, 14, 5, 6, 9, 10, 1), in $T_{16}\langle 1, 3, 4; 9\rangle$ the hamiltonian circuit (1, 2, 3, 4, 5, 8, 11, 14, 15, 6, 9, 12, 13, 16, 7, 10, 1), in $T_{17}\langle 1, 3, 4; 9\rangle$ the circuit (1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 17, 8, 9, 10, 1), in $T_{20}\langle 1, 3, 4; 9\rangle$ the circuit (1, 4, 5, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 11, 2, 3, 6, 7, 10, 1), and in $T_{23}\langle 1, 3, 4; 9\rangle$ the circuit (1, 4, 5, 8, 9, 12, 13, 16, 17, 18, 21, 22, 23, 14, 15, 19, 20, 11, 2, 3, 6, 7, 10, 1) (see Figs. 34-40).

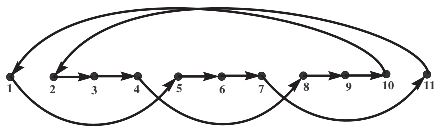

Fig. 34.

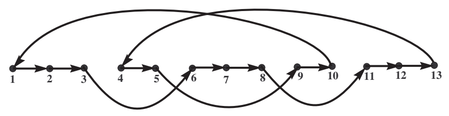

Fig. 35.

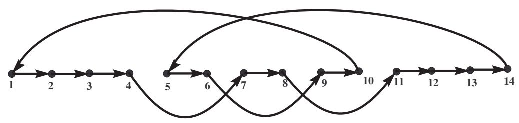

Fig. 36.

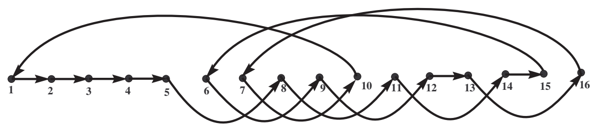

Fig. 37.

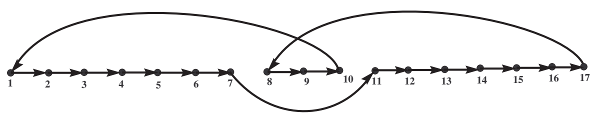

Fig. 38.

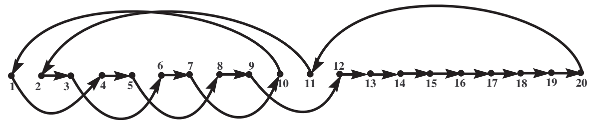

Fig. 39.

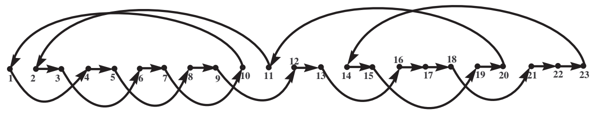

Fig. 40.

Starting from the above values of $n \in \{10, 11, 13, 14, 16, 17, 20, 23\}$, we succesively extend a hamiltonian circuit in $T_n\langle 1, 3, 4; 9\rangle$ containing the arc $(n-2,\, n-1)$ to a hamiltonian circuit in $T_{n+8}\langle 1, 3, 4; 9\rangle$ with this same property by replacing the arc $(n-2,\, n-1)$ with the path

$$(n-2,\, n+1,\, n+2,\, n+3,\, n+4,\, n+5,\, \underline{n+6,\, n+7},\, n+8,\, n-1)$$

Since 10, 11, 13, 14, 16, 17, 20, 23 are representatives in each of the various rest classes modulo 8, it follows that $T_n\langle 1, 3, 4; 9\rangle$ is hamiltonian for $n = 10, 11, 13, 14$ and $n \geq 16$. This together with Claim 1 shows that $T_n\langle 1, 3, 4; 9\rangle$ is hamiltonian for all $n \neq 15$. □

4 Toeplitz graphs $T_n\langle 1, 3, 4; t\rangle$ with $t \geq 10$

Theorem 12. $T_n\langle 1, 3, 4; t\rangle$, $t \geq 10$, *is hamiltonian for all* n.

Proof. First we remark that, for any vertex a and $b = a + 5 + 4r$; $r \in \mathbb{N}$, of $T_n\langle 1, 3, 4; t\rangle$, there exists a path $M_{a\to b}$ from a to b namely

$$(a,\, a+1,\, a+4,\, a+5,\, a+8,\, a+9,\, \ldots,\, b-5,\, b-4,\, b-1,\, b)$$

(see Fig. 41).

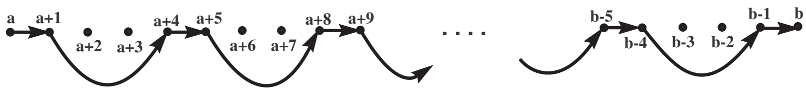

Fig. 41.

Claim 1. $T_{t+3}\langle 1,3,4;t\rangle$ is hamiltonian.

Indeed $T_{t+3}\langle 1,3,4;t\rangle$ has one of the following hamiltonian circuits, depending upon t.
(i) If $t+3 \cong 0\, mod\, 4$, then a hamiltonian circuit is
(1, 2, 5, 6, 7, 10, $M_{11\to t+3}$, 3, 4, 8, 9, $M_{13\to t+1}$, 1), see Fig. 42.

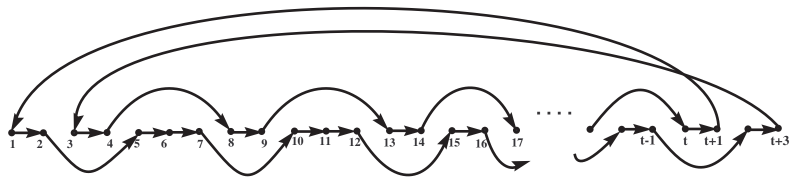

Fig. 42.

(ii) If $t+3 \cong 1\, mod\, 4$, then a hamiltonian circuit is
(1, 2, 6, 7, 8, $M_{12\to t+3}$, 3, 4, 5, 9, $M_{10\to t+1}$, 1), see Fig. 43.

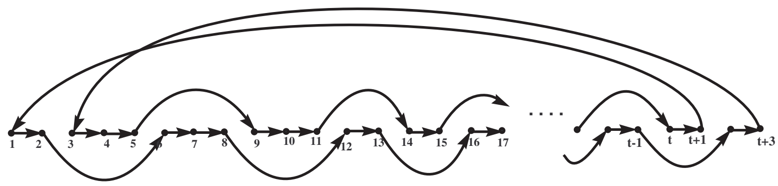

Fig. 43.

(iii) If $t+3 \cong 2\, mod\, 4$, then a hamiltonian circuit is $(M_{1\to t+3},\, M_{3\to t+1},\, 1)$, see Fig. 44.

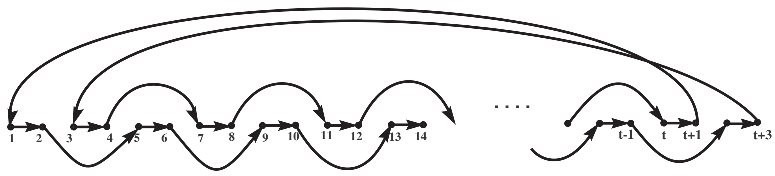

Fig. 44.

(iv) If $t+3 \cong 3\, mod\, 4$, then a hamiltonian circuit is $(1,\, 2,\, 5,\, 6,\, M_{10\to t+3},$ $3,\, 4,\, 7,\, M_{8\to t+1},\, 1)$, see Fig. 45.

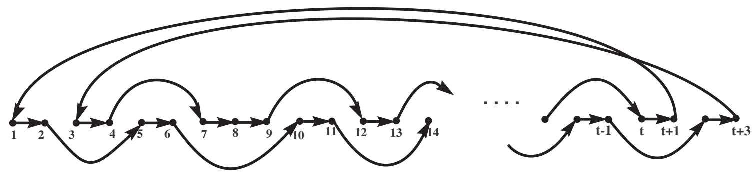

Fig. 45.

Claim 2. For $n \in \{t+1,\, t+2,\, t+4,\, t+5,\, \dots,\, 2t-1,\, 2t+2\}$, $T_n\langle 1,3,4;t\rangle$ has a hamiltonian circuit containing the arc $(n-2,\, n-1)$.

Indeed $T_{t+1}\langle 1,3,4;t\rangle$ has a hamiltonian circuit

$$T_{t+1}\langle 1;t\rangle = (1,\, 2,\, \dots,\, \underline{t-1,\, t},\, t+1,\, 1).$$

$T_{t+2}\langle 1, 3, 4; t\rangle$ has one of the following hamiltonian circuits, depending upon t.
(i) If $t + 2 \cong 0\, mod\, 4$, then a hamiltonian circuit is

(1, 5, 6, $M_{10\to t-1}$, $t + 2$, 2, 3, 4, 7, $M_{8\to t-3}$, $\underline{t,\, t + 1}$, 1), see Fig. 46.

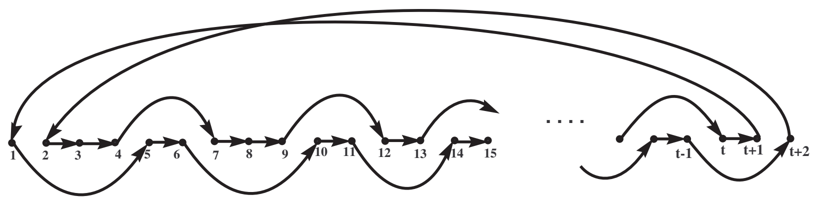

Fig. 46.

(ii) If $t + 2 \cong 1\, mod\, 4$, then a hamiltonian circuit is

(1, 5, 6, 7, $M_{11\to t-1}$, $t + 2$, 2, 3, 4, 8, $M_{9\to t-3}$, $\underline{t,\, t + 1}$, 1), see Fig. 47.

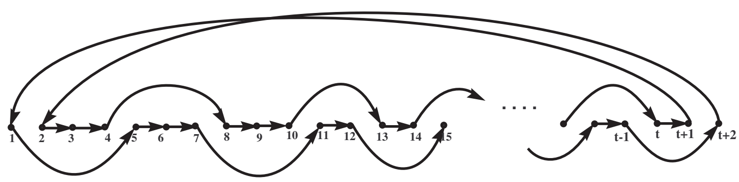

Fig. 47.

(*iii*) If $t+2 \cong 2\, mod\, 4$, then a hamiltonian circuit is (1, $M_{4\to t-1}$, $t+2$, $M_{2\to t-3}$, $\underline{t,\, t+1}$, 1), see Fig. 48.

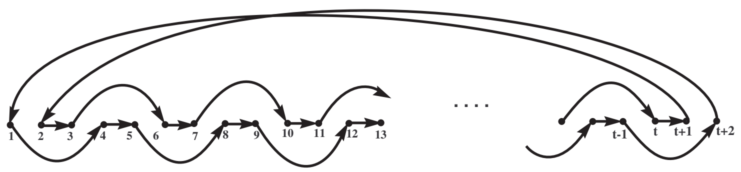

Fig. 48.

(*iv*) If $t+2 \cong 3\, mod\, 4$, then a hamiltonian circuit is (1, 4, 5, $M_{9\to t-1}$, $t+2$, 2, 3, 6, $M_{7\to t-3}$, $\underline{t,\, t+1}$, 1), see Fig. 49.

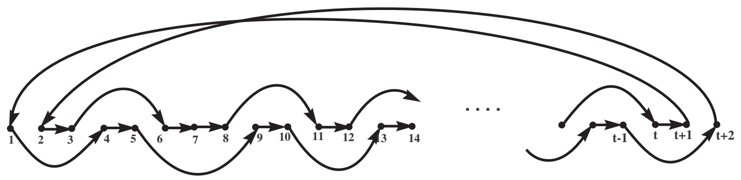

Fig. 49.

Now for every $n \in \{t+4,\, t+5,\, \ldots,\, 2t-5,\, 2t-4\}$, $T_n\langle 1,3,4;t\rangle$ has one of the following hamiltonian circuits, depending upon t and n.

(*i*) If $2t-n \cong 0\, mod\, 4$, then a hamiltonian circuit is

(1, 2, …, $n-t-3$, $M_{n-t-2\to t+3}$, $t+4$, $t+5$, …, $\underline{n-2,\, n-1}$, n, $M_{n-t\to t+1}$, 1), see Fig. 50.

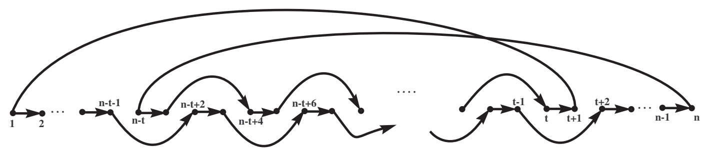

Fig. 50.

(*ii*) If $2t - n \cong 1\, mod\, 4$, then a hamiltonian circuit is $(1, 2, \ldots, n-t-1, n-t+2, M_{n-t+3 \to t+3}, t+4, t+5, \ldots, \underline{n-2, n-1}, n, n-t, n-t+1, M_{n-t+5 \to t+1}, 1)$, see Fig. 51.

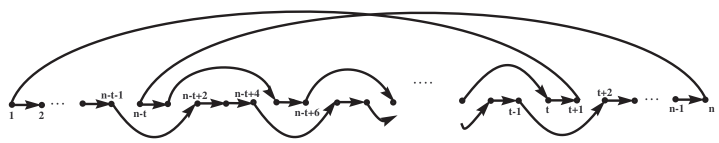

Fig. 51.

(*iii*) If $2t - n \cong 2\, mod\, 4$, then a hamiltonian circuit is $(1, 2, \ldots, n-t-1, n-t+3, M_{n-t+4 \to t+3}, t+4, t+5, \ldots, \underline{n-2, n-1}, n, n-t, n-t+1, n-t+2, M_{n-t+6 \to t+1}, 1)$, see Fig. 52.

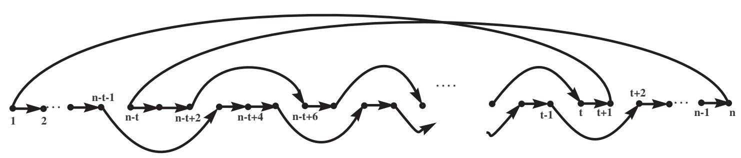

Fig. 52.

(iv) If $2t - n \cong 3\, mod\, 4$, then a hamiltonian circuit is

$(1,\ 2,\ \ldots,\ n-t-1,\ n-t+3,\ n-t+4,\ n-t+5,\ M_{n-t+9\to t+3},\ t+4,\ t+5,\ \ldots,\ \underline{n-2,\ n-1},\ n,\ n-t,\ n-t+1,\ n-t+2,\ n-t+6,\ M_{n-t+7\to t+1},\ 1)$, see Fig. 53.

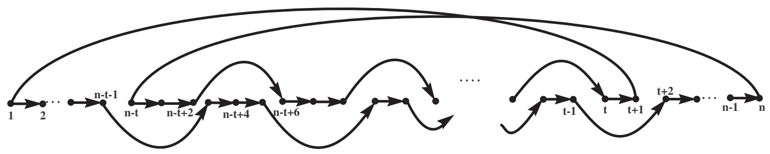

Fig. 53.

$T_{2t-3}\langle 1,3,4;t\rangle$ has one of the following hamiltonian circuits, depending upon t.

(i) If $t \cong 0\, mod\, 4$, then a hamiltonian circuit is

$(1,\ 2,\ \ldots,\ t-5,\ t-2,\ t-1,\ t+2,\ t+3,\ M_{t+7\to 2t-8},\ \underline{2t-5,\ 2t-4},\ t-4,\ t,\ t+4,\ M_{t+5\to 2t-6},\ 2t-3,\ t-3,\ t+1,\ 1)$, see Fig. 54.

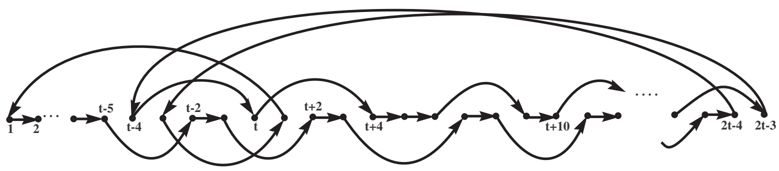

Fig. 54.

(ii) If $t \cong 1\, mod\, 4$, then a hamiltonian circuit is

$(1,\, 2,\, \ldots,\, t-5,\, t-2,\, t-1,\, t+2,\, t+3,\, t+7,\, M_{t+8\to 2t-8},\, \underline{2t-5,\, 2t-4},\, t-4,\, t,\, t+4,\, t+5,\, t+6,\, M_{t+10\to 2t-6},\, 2t-3,\, t-3,\, t+1,\, 1)$, see Fig. 55.

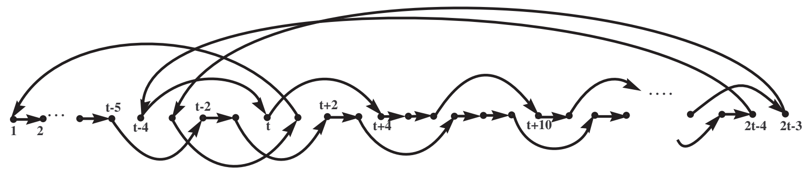

Fig. 55.

(iii) If $t \cong 2\, mod\, 4$, then a hamiltonian circuit is

$(1,\, 2,\, \ldots,\, t{-}5,\, t{-}2,\, t{-}1,\, t{+}2,\, M_{t+5\to 2t-8},\, \underline{2t-5,\, 2t-4},\, t{-}4,\, t,\, M_{t+3\to 2t-6},\, 2t-3,\, t-3,\, t+1,\, 1)$, see Fig. 56.

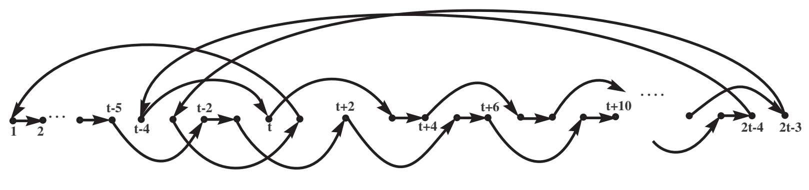

Fig. 56.

(iv) If $t \cong 3\,mod\,4$, then a hamiltonian circuit is

$(1,\, 2,\, \ldots,\, t-7,\, M_{t-6\to 2t-8},\, \underline{2t-5,\, 2t-4},\, t-4,\, t,\, M_{t+4\to 2t-6},\, 2t-3,\, t-3,\, t+1,\, 1)$, see Fig. 57.

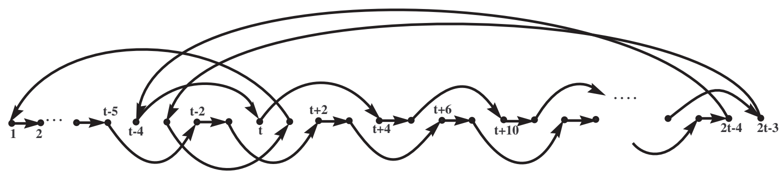

Fig. 57.

$T_{2t-2}\langle 1, 3, 4; t\rangle$, has one of the following hamiltonian circuit depending upon t.

(i) If $t \cong 0\,mod\,4$, then a hamiltonian circuit is

$(1,\, 2,\, \ldots,\, t-4,\, t-1,\, t+2,\, t+3,\, t+7,\, M_{t+8\to 2t-7},\, \overline{2t-4,\, 2t-3},\, t-3,\, t,\, t+4,\, t+5,\, t+6,\, M_{t+10\to 2t-5},\, 2t-2,\, t-2,\, t+1,\, 1)$, see Fig. 58.

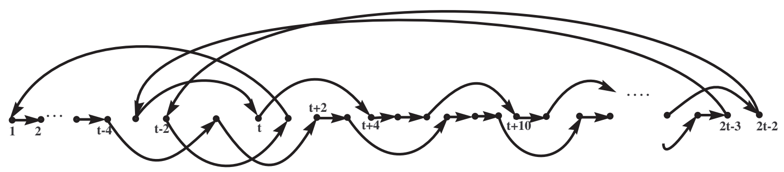

Fig. 58.

(ii) If $t \cong 1 \, mod \, 4$, then a hamiltonian circuit is

$(1, 2, \ldots, t-4, t-1, t+2, M_{t+5\rightarrow 2t-7}, \underline{2t-4, 2t-3}, t-3, t, M_{t+3\rightarrow 2t-5}, 2t-2, t-2, t+1, 1)$, see Fig. 59.

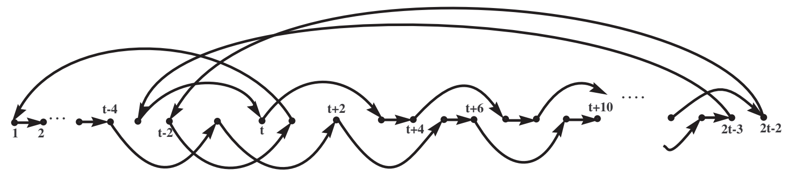

Fig. 59.

(iii) If $t \cong 2 \, mod \, 4$, then a hamiltonian circuit is

$(1, 2, \ldots, t-4, t-1, M_{t+2\rightarrow 2t-7}, \underline{2t-4, 2t-3}, t-3, t, M_{t+4\rightarrow 2t-5}, 2t-2, t-2, t+1, 1)$, see Fig. 60.

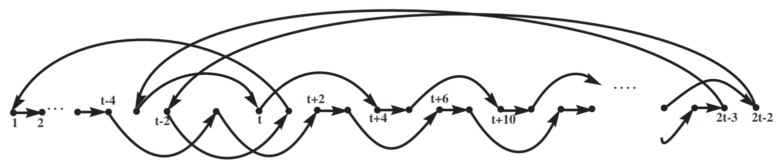

Fig. 60.

(iv) If $t \cong 3\, mod\, 4$, then a hamiltonian circuit is

$(1,\ 2,\ \ldots,\ t-4,\ t-1,\ t+2,\ t+3,\ M_{t+7\to 2t-7},\ \underline{2t-4,\ 2t-3},\ t-3,\ t,\ t+4,\ M_{t+5\to 2t-5},\ 2t-2,\ t-2,\ t+1,\ 1)$, see Fig. 61.

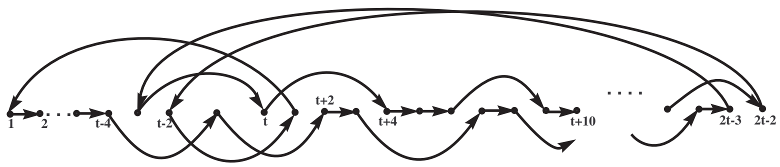

Fig. 61.

$T_{2t-1}\langle 1, 3, 4; t\rangle$ has a hamiltonian circuit

$(1,\ 2,\ \ldots,\ t-2,\ t+2,\ t+3,\ \ldots,\ \underline{2t-3,\ 2t-2},\ 2t-1,\ t-1,\ t,\ t+1,\ 1)$, see Fig. 62.

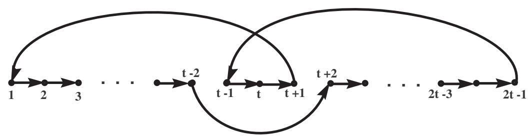

Fig. 62.

$T_{2t+2}\langle 1,3,4;t\rangle$ has one of the following hamiltonian circuit depending upon t.

(i) If $t \cong 0\, mod\, 4$, then a hamiltonian circuit is (1, 4, 5, 6, 9, $M_{10\to t-1}$, $t+3$, $t+4$, ..., $\underline{2t,\ 2t+1}$, $2t+2$, $t+2$, 2, 3, 7, 8, $M_{12\to t+1}$,1), see Fig. 63.

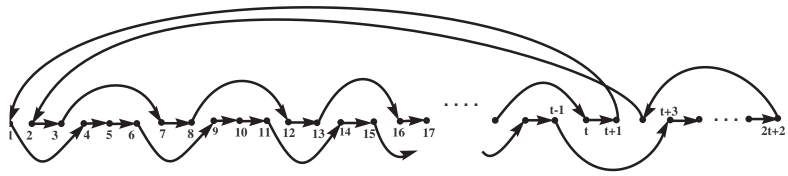

Fig. 63.

(ii) If $t \cong 1\, mod\, 4$, then a hamiltonian circuit is (1, 4, 5, 6, 10, $M_{11\to t-1}$, $t+3$, $t+4$, ..., $\underline{2t,\ 2t+1}$, $2t+2$, $t+2$, 2, 3, 7, 8, 9, $M_{13\to t+1}$, 1), see Fig. 64.

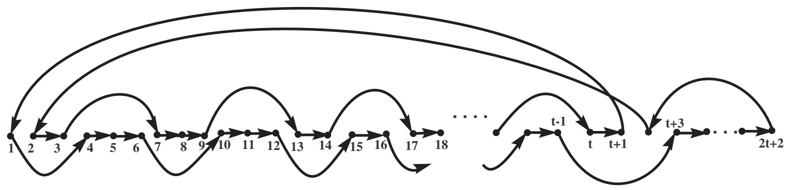

Fig. 64.

(iii) If $t \cong 2\, mod\, 4$, then a hamiltonian circuit is

(1, $M_{4\to t-1}$, $t+3$, $t+4$, ..., $\underline{2t,\ 2t+1}$, $2t+2$, $t+2$, $M_{2\to t+1}$, 1), see Fig. 65.

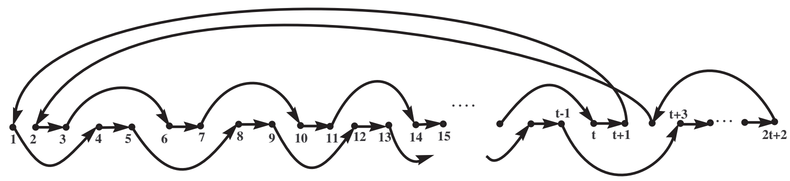

Fig. 65.

(iv) If $t \cong 3\, mod\, 4$, then a hamiltonian circuit is

(1, 4, 5, $M_{9\to t-1}$, $t+3$, $t+4$, $\ldots$, $\underline{2t,\ 2t+1}$, $2t+2$, $t+2$, 2, 3, 6, $M_{7\to t+1}$, 1), see Fig. 66.

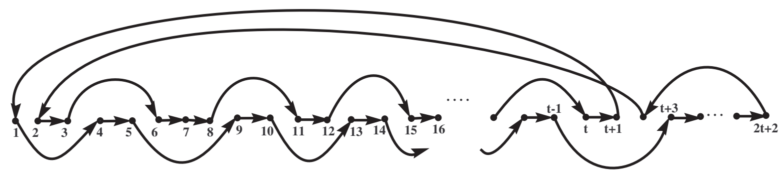

Fig. 66.

Starting from the above values of $n \in \{t+1,\ t+2,\ t+4,\ t+5,\ \ldots,\ 2t-1,\ 2t+2\}$, we can extend a hamiltonian circuit in $T_n\langle\ 1,\ 3,\ 4;\ t\rangle$ containing the arc $(n-2,\ n-1)$ to a hamiltonian circuit in $T_{n+t-1}\langle\ 1,\ 3,\ 4;\ t\rangle$ with the same property by replacing the arc $(n-2,\ n-1)$ with the path

$$(n-2,\ n+1,\ n+2,\ n+3,\ \ldots,\ \underline{n+t-3,\ n+t-2},\ n+t-1,\ n-1).$$

Since $t+1$, $t+2$, $t+4$, $t+5$, $\ldots$, $2t-1$, $2t+2$ are representatives in each of the various rest classes modulo $t-1$, it follows that $T_n\langle 1,3,4;t\rangle$ is hamiltonian for $n = t+1$, $t+2$ and all $n \geq t+4$. This together with Claim 1 shows that $T_n\langle 1,3,4;t\rangle$ is hamiltonian for all n. $\square$

Conjectures:

1. $T_n\langle 1,3,4;t\rangle$ is non-hamiltonian for $n \cong 1,2,5 \ mod\, 6$ such that $n \notin \{5,7\}$.

2. $T_n\langle 1,3,4;t\rangle$ is non-hamiltonian for $n \in \{12,13,16,19,22\}$.

3. $T_{15}\langle 1,3,4;9\rangle$ is non-hamiltonian.

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