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Electron whistler mode instability in an inhomogeneous thermal plasma in the presence of an inhomogeneous beam of suprathermal electrons

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The excitation of the whistler mode waves propagating obliquely to the constant and uniform magnetic field in a warm and inhomogeneous plasma in the presence of an inhomogeneous beam of suprathermal electrons is studied. The full dispersion relation including electromagnetic effects is derived. In the electrostatic limit the expression for the growth rate is given. It is found that the inhomogeneities in both beam and plasma number densities affect the growth rates of the instabilities.

1. Introduction

The excitation of whistler mode instabilities ($\omega_{ci} \ll \omega < \omega_{ce}$) due to an electron beam propagating through a plasma has been studied for many years (e.g. Tataronis & Crawford 1970; Brinca 1972; Ossakow, Otto & Haber 1973; Hashimoto & Kimura 1977; Kumagai, Hashimoto & Kimura 1980). In the above studies the beam–plasma system was assumed to be homogeneous. In a recent paper Freund, Dillenburg & Wu (1982) have investigated the excitation of whistler waves in the case of an inhomogeneous suprathermal electron beam interacting with a cold homogeneous plasma. However, in many real situations both the beam and plasma can be inhomogeneous and hot. In this paper we consider the case of a warm inhomogeneous plasma penetrated by an inhomogeneous suprathermal beam of electrons. The direction of propagation of the beam is taken along the externally applied magnetic field which is assumed to be constant and uniform. The effect of the inhomogeneity appears through the gradient in the number densities of both beam and plasma.

The paper is organized as follows. In §2 we compute the elements of the dielectric tensor for the beam and the plasma. In §3 the general dispersion relation including electromagnetic effects is derived. The expression of the growth rate in the electrostatic limit is given. In §4 the results of a numerical analysis are presented and discussed.

2. Dielectric tensor

In this section we proceed to determine the various elements of the dielectric tensor for the beam and the plasma. Since we are specifically interested in the electron whistler mode ($\omega_{ci} \ll \omega < \omega_{ce}$) instability due to Čerenkov interactions, only $m = 0$ mode is of relevance to our computations. It may be noted that $\omega_{c\alpha} = e_\alpha B_0/m_\alpha c$ is the ion or electron gyrofrequency. The inhomogeneity is considered weak and the density gradient is taken in the y direction. The externally applied magnetic field $\mathbf{B}_0$ is in the z direction and we shall be considering propagation in the (x, z) plane.

The general expression for the dielectric tensor for an inhomogeneous thermal plasma, with a streaming velocity v_z (along the direction of the externally applied magnetic field), has been given by Mikhailovskii (1967) and has the form

$$\begin{aligned}\epsilon_{lm}(\mathbf{k}, \omega, y) = \delta_{lm} + \sum_\alpha \frac{4\pi e_\alpha}{m_\alpha \omega}\Bigg[2\pi \int dv_{\perp 0,\alpha}^2\, dv_{z\alpha,0}\left\{\Phi_\alpha Q_{lm} - \frac{1}{\omega_{c\alpha}}\frac{\partial \Phi_\alpha}{\partial y} P_{lm}\right\} \\ - \frac{\delta_{mx}}{\omega\omega_{c\alpha}}\left\{\delta_{lz}\int v_{z\alpha,0}\frac{\partial f_{0\alpha}}{\partial y} d\mathbf{v}_{\alpha,0} - \delta_{lx}\int \frac{v_{\perp 0,\alpha}^2}{2\omega_{c\alpha}}\frac{\partial^2 f_{0\alpha}}{\partial y^2} d\mathbf{v}_\alpha\right\} \\ + \frac{\delta_{mz}}{\omega}\left\{\int v_{z\alpha,0}\left(\frac{\partial f_{0\alpha}}{\partial v_z} - m v_z \frac{\partial f_{0\alpha}}{\partial \epsilon_\alpha}\right) d\mathbf{v}_{\alpha,0}\,\delta_{lz}\right\} \\ - \delta_{lx}\frac{\partial}{\partial y}\int \frac{v_{\perp\alpha}^2}{2\omega_{c\alpha}}\left(\frac{\partial f_{0\alpha}}{\partial v_z} - m v_z\frac{\partial f_{0\alpha}}{\partial \epsilon_\alpha}\right) d\mathbf{v}_{\alpha,0}\Bigg] \end{aligned} \tag{1}$$

where

$$\Phi_\alpha = m_\alpha \frac{\partial f_{0\alpha}}{\partial \epsilon_\alpha}\left(1 - \frac{k_z v_z}{\omega}\right) - \frac{k_x}{\omega\omega_{c\alpha}}\frac{\partial f_{0\alpha}}{\partial y} + \frac{k_z}{\omega}\frac{\partial f_{0\alpha}}{\partial v_z}, \tag{2}$$

$$Q_{lm} = q_l^* q_m/(\omega - k_z v_{z\alpha}), \tag{3}$$

$\mathbf{q}^*$ is the complex conjugate of $\mathbf{q}$, and

$$\mathbf{q} = (0, i v_{\perp\alpha} J_0', v_{z\alpha} J_0); \tag{4}$$

$$P_{lm} = p_l q_m/(\omega - k_z v_{z\alpha}) \tag{5}$$

and

$$\mathbf{p} = (v_{\perp 0\alpha}^2 (J_0 + J_0''), 0, 0), \tag{6}$$

where $J_0 = J_0(k_x v_{\perp\alpha}/\omega_{c\alpha})$ is the Bessel function and J_0' and J_0'' are its first and second derivatives respectively. m_α, ϵ_α are the mass, and total energy of particles of species α respectively, and $d\mathbf{v}_{0\alpha} = dv_{\perp 0\alpha}^2\, dv_{z0\alpha}\, d\phi_0$.

We begin by computing the dielectric tensors for the beam. The beam is taken to be one consisting of electrons only and having a Maxwellian distribution of the form

$$f_b = n_b(y)\left(\frac{2\pi T_{\perp b}}{m}\right)\left(\frac{m}{2\pi T_{\parallel b}}\right)^{\frac{1}{2}} \exp\left\{-\frac{m v_{\perp 0}^2}{2T_{\perp b}} - \frac{m(v_{z0} - v_z)^2}{2T_{\parallel b}}\right\} \tag{7}$$

where n_b is the beam density and $T_{\perp b}$ and $T_{\parallel b}$ are perpendicular and parallel temperatures respectively. In order to carry out integration over dv_{z0} we make use of the formula (Mikhailovskii 1967)

$$\left(\frac{m}{2\pi T_\parallel}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{dv_{z0}\exp\{-m(v_{z0} - v_z)^2/T_\parallel\}}{\omega - k_z v_{z0}} = -\frac{1}{k_z v_{\text{th}z}} Z\left(\frac{\omega - k_z v_z}{k_z v_{\text{th}z}}\right) \tag{8}$$

where Z is the Fried–Conte dispersion function (Fried & Conte 1961) and $v_{\text{th}z} = (2T_\parallel/m)^{\frac{1}{2}}$.

In the integration over transverse velocities we make use of the following two formulae. Firstly (Gradshtein & Ryzhik 1966)

$$\int_0^\infty \exp(-\sigma^2x^2)\,J_s(\alpha x)\,J_s(\beta x)\,dx = \frac{1}{2\sigma^2}\exp\left(-\frac{\alpha^2+\beta^2}{4\sigma^2}\right)I_s\left(\frac{\alpha\beta}{2\sigma^2}\right), \tag{9}$$

where I_s is the Bessel function of imaginary argument. It may be noted that we consider the $s = 0$ mode only. Secondly

$$\int_0^\infty x^{\lambda-1}\exp(-\alpha x^2)\,J_s(\beta x)\,J_t(\beta x)\,dx = 2^{-s-t-1}\alpha^{-\frac{1}{2}(s+t+\lambda)}\beta^{s+t}\frac{\Gamma(\frac{1}{2}\lambda+\frac{1}{2}s+\frac{1}{2}t)}{\Gamma(s+1)\,\Gamma(t+1)}$$
$$\times\,{}_3F_3\left(\frac{s}{2}+\frac{t}{2}+\frac{1}{2};\frac{s}{2}+\frac{t}{2}+1;\frac{s+t+v}{2};\,s+1,t+1,s+t+1;\,-\frac{\beta^2}{\alpha}\right), \tag{10}$$

where ${}_3F_3$ is the hypergeometric series. In our specific case where the $s = 0$ mode is investigated, (10) simplifies and the ${}_3F_3$ hypergeometric function is replaced by ${}_1F_1$, the confluent hypergeometric function.

After performing the necessary integration we obtain expressions for the dielectric tensors of the beam:

$$\epsilon^b_{xx} = 1,$$

$$\epsilon^b_{xy} = \frac{6ia\theta d_2\xi\psi_b}{\alpha^2}\left\{\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d^{\frac{1}{2}}\xi}Z_b-\frac{d_2}{d_1}\right\}F_b,$$

$$\epsilon^b_{xz} = \frac{-2a\theta d_2\psi_b}{d_1\alpha^2}\left[\left\{\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\left(1+\frac{d_1}{d_2}\right)-\frac{\alpha}{\eta d^{\frac{1}{2}}\xi}+\frac{d_1}{d_2}\frac{\alpha}{\eta d_1^{\frac{1}{2}}\xi}\right.\right.$$
$$\left.\left.\times\frac{(\alpha-\eta b^{\frac{1}{2}}\xi)}{\eta d_1^{\frac{1}{2}}\xi}Z_b\right\}F_b-\left(\frac{b}{d_1}\right)^{\frac{1}{2}}\left(\frac{d_1}{d_2}+1\right)\right],$$

$$\epsilon^b_{yx} = 0,$$

$$\epsilon^b_{yy} = 1+\frac{a\theta d_2^2\xi^2}{d_1\alpha^2}\left[\left\{\frac{d_1}{d_2}\frac{(\alpha-\eta b^{\frac{1}{2}}\xi)}{\eta d_1^{\frac{1}{2}}\xi}-\frac{\psi_b d_1^{\frac{1}{2}}}{2\eta}\right\}Z_b-1\right]F_b,$$

$$\epsilon^b_{yz} = \frac{-2ia\theta d_2\xi}{d_1\alpha^2}\left[\frac{\alpha}{\eta d_1^{\frac{1}{2}}\xi}\left\{\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\frac{d_2}{d_1}-\frac{\psi_b d_1^{\frac{1}{2}}}{2\eta}\right\}Z_b\right.$$
$$\left.+\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\left(\frac{d_2}{d_1}+1\right)-\frac{\psi_b d_1^{\frac{1}{2}}}{2\eta}-\frac{\alpha}{\eta d_1^{\frac{1}{2}}\xi}\right]F_b,$$

$$\epsilon^b_{zx} = \frac{2\pi^{\frac{1}{2}}a\theta b^{\frac{1}{2}}\psi_b}{\alpha^2},$$

$$\epsilon^b_{zy} = -\epsilon^b_{yz},$$

$$\epsilon^b_{zz} = 1+\frac{4a\theta}{\alpha^2}\left\{\left[-Z_b\left\{\left[\frac{b}{d_1}+\frac{2(\alpha-\eta b^{\frac{1}{2}}\xi)}{\eta d_1^{\frac{1}{2}}\xi}+\left(\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\right)^2\right]\right.\right.\right.$$
$$\times\left(\frac{\alpha}{\eta d_1^{\frac{1}{2}}\xi}-\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}-\frac{\psi_b d_1^{\frac{1}{2}}}{2\eta}\right)+\frac{\alpha+2\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\left(\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\right)^2$$
$$\left.+\frac{b}{d_1}\left(\left(\frac{b}{d_1}\right)^{\frac{1}{2}}+\frac{3(\alpha-\eta b^{\frac{1}{2}}\xi)}{\eta d_1^{\frac{1}{2}}\xi}\right)\right\}-\frac{\alpha+\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\left(\frac{\alpha}{\eta d_1^{\frac{1}{2}}\xi}-\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}\frac{d_1}{d_2}\right.$$
$$\left.\left.\left.-\frac{\psi_b d_1^{\frac{1}{2}}}{2\eta}\right)+\frac{(\alpha+2\eta b^{\frac{1}{2}}\xi)}{\eta d_1^{\frac{1}{2}}\xi}\frac{\alpha-\eta b^{\frac{1}{2}}\xi}{\eta d_1^{\frac{1}{2}}\xi}+\frac{1}{2}+\frac{3b}{d_1}\right]\Lambda_b+\left(1+\frac{d_2}{d_1}\right)\frac{b}{d_1}+\frac{1}{2}\right\}. \tag{11}$$

In equations (11) all variables have been normalized by the parameters of the electron components of the bulk plasma in the following manner:

$$\frac{\omega}{\omega_{ce}} = \alpha, \quad \frac{k_z}{k_x} = \eta, \quad \frac{k_x v_{\text{th}\perp e}}{\omega_{ce}} = \xi,$$

$$\frac{\omega_{pe}^2}{\omega_{ce}^2} = a, \quad \frac{\omega_b^2}{\omega_{ce}^2} = a\theta, \quad \theta = \frac{n_b}{n_e}(\ll 1),$$

$$\frac{T_{\parallel b}}{T_{\perp e}} = d_1, \quad \frac{T_{\perp b}}{T_{\perp e}} = d_2, \quad \frac{v_z^2}{v_{\text{th}\perp e}} = b,$$

$$v_{\text{th}\perp e} = \left(\frac{2T_{\perp e}}{m_e}\right)^{\frac{1}{2}}, \quad \frac{v_{\text{th}\perp e}}{L_b \omega_{ce}} = \psi_b,$$

where $L_b = n_b(\partial n_b/\partial y)^{-1}$ is the scale length of the beam inhomogeneity. n_e, $T_{\perp e}$ $T_{\parallel e}$ are number density and perpendicular and parallel thermal spreads of the electron component of the bulk plasma. $Z_b = Z(\alpha - \eta b^{\frac{1}{2}}\xi)/\eta d_1^{\frac{1}{2}}\xi$ is the Fried–Conte plasma dispersion function for the beam. $F_b = {}_1F_1(\frac{3}{2}; 2; -\xi^2 d_2)$ is the confluent hypergeometric function with beam parameters.

In order to obtain expressions for the dielectric tensor elements of the bulk plasma we put $v_z = 0$ in (2). The distribution function of the plasma is given by

$$f_{0\alpha} = \sum_\alpha n_{0\alpha}(y)\left(\frac{m_\alpha}{2\pi T_{\perp\alpha}}\right)\left(\frac{m_\alpha}{2\pi T_{\parallel\alpha}}\right)^{\frac{1}{2}} \exp\left(-\frac{m_\alpha v_{\perp 0\alpha}^2}{2T_{\perp\alpha}} - \frac{m_\alpha v_{\perp 0\alpha}^2}{2T_{z\alpha}^2}\right)$$

where summation is over species (electrons and ions).

Using (8) for the case $v_z = 0$ and (9) and (10) we can obtain expressions for the dielectric tensor elements of the plasma,

$$\epsilon_{xx}^p = 1,$$

$$\epsilon_{xy}^p = \frac{6ia\xi\psi_p}{\alpha^2}\left(\frac{\alpha}{\eta g^{\frac{1}{2}}\xi}Z_e - \frac{1}{g}\right)F_e + \frac{6ia\xi\psi_p h_2}{\alpha^2}\left(\frac{\alpha}{\eta\xi h_1^{\frac{1}{2}}\beta}\right)Z_i - \frac{h_2}{h_1}\Big) F_i,$$

$$\epsilon_{xz}^p = \frac{-2a\omega_p g^{\frac{1}{2}}}{\alpha^2}\left\{\frac{\alpha}{\eta g^{\frac{1}{2}}\xi} + \left(\frac{\alpha}{\eta g^{\frac{1}{2}}\xi}\right)^2 Z_e\right\}F_e + \frac{2a(\beta h_1)^{\frac{1}{2}}\psi_p}{\alpha^2}\left\{\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}} + \left(\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}\right)^2 Z_i\right\}F_i,$$

$$\epsilon_{yx}^p = 0,$$

$$\epsilon_{yy}^p = 1 + \frac{2a\xi^2}{g\alpha^2}\left\{\left(g\frac{\alpha}{\eta\xi g^{\frac{1}{2}}} - \frac{\psi_p g^{\frac{1}{2}}}{2\eta}\right)Z_e - 1\right\}F_e + \frac{2ah_2^2\xi^2}{\alpha^2 h_1}\left\{\left(\frac{h_1}{h_2}\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}} + \frac{\psi_p h^{\frac{1}{2}}}{2\eta\beta^{\frac{1}{2}}}\right)Z_i - 1\right\}F_i,$$

$$\epsilon_{yz}^p = \frac{-2ia\xi}{g^{\frac{1}{2}}\alpha^2}\left\{\frac{\alpha}{\eta\xi g^{\frac{1}{2}}}\left(g\frac{\alpha}{\eta\xi g^{\frac{1}{2}}} - \frac{\psi_p g^{\frac{1}{2}}}{2\eta}\right)Z_e + \frac{g\alpha}{\eta\xi g^{\frac{1}{2}}} - \frac{\psi_p g^{\frac{1}{2}}}{2\eta}\right\}F_e$$
$$+ \frac{2ia\beta^{\frac{1}{2}}h_2\xi}{h^{\frac{1}{2}}\alpha^2}\left\{\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}\left(\frac{h_1}{h_2}\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}} + \frac{\psi_p h_1^{\frac{1}{2}}}{2\eta\beta^{\frac{1}{2}}}\right)Z_i + \frac{h_1}{h_2}\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}} + \frac{\psi_p h_1^{\frac{1}{2}}}{2\eta\beta^{\frac{1}{2}}}\right\}F_i,$$

$$\epsilon_{zx}^p = 0,$$

$$\epsilon_{zy}^p = -\epsilon_{yz}^p,$$

$$\epsilon_{zz}^p = 1 + \frac{4a}{\alpha^2}\left\{\left[-Z_e\left\{-\left[\left(\frac{2\alpha}{\eta\xi g^{\frac{1}{2}}}\right) + \left(\frac{\alpha}{\eta\xi g^{\frac{1}{2}}}\right)^2\right]\frac{\psi_p g^{\frac{1}{2}}}{2\eta} + \left(\frac{\alpha}{\eta\xi g^{\frac{1}{2}}}\right)^3\right\}\right.\right.$$
$$\left.\left. - \left(\frac{\alpha}{\eta\xi g^{\frac{1}{2}}} - \frac{g\alpha}{\eta\xi g^{\frac{1}{2}}} - \frac{\psi_p g^{\frac{1}{2}}}{2\eta}\right)\frac{\alpha}{\eta\xi g^{\frac{1}{2}}} + \left(\frac{\alpha}{\eta\xi g^{\frac{1}{2}}}\right)^2\right]\Lambda_e + \frac{1}{2}\right\}$$

$$+\frac{4\alpha\beta}{\alpha^2}\left\{\left[-Z_i\left\{\left[\frac{2\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}+\left(\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}\right)^2\right]\frac{\psi_p h_1^{\frac{1}{2}}}{2\eta\beta^{\frac{1}{2}}}+\left(\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}\right)^3\right\}\right.\right.$$
$$\left.\left.-\left(\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}-\frac{h_1}{h_2}\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}\right)\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}+\left(\frac{\alpha}{\eta\xi(h_1\beta)^{\frac{1}{2}}}\right)^2\right]\Lambda_i+\tfrac{1}{2}\right\}.$$

In (12) all parameters have been normalized with electron plasma parameters

$$\frac{T_{\parallel e}}{T_{\perp e}}=g,\quad \frac{T_{\parallel i}}{T_{\perp e}}=h_1,\quad \frac{T_{\parallel i}}{T_{\perp e}}=h_2,\quad \frac{v_{\text{th}\perp}}{L_p\omega_{ce}}=\psi_p,$$

where $1/L_p=(1/n_{i,e})\,\partial n_{i,e}/\partial y$ is the scale length of the plasma density inhomogeneity.

$$F_e=F(\tfrac{3}{2}:2;\,-\xi^2),$$
$$F_i=F(\tfrac{3}{2};2;\,-\xi^2h_2/\beta),$$

where F_e and F_i are the confluent hypergeometric series.

$$\Lambda_e=\exp(-\xi^2)\,I_0(\xi^2),$$
$$\Lambda_i=\exp\left(-\frac{\xi^2h_2}{\beta}\right)I_0\left(\frac{\xi^2h_2}{\beta}\right).$$

3. Dispersion relation

The general dispersion relation including electromagnetic effects is given by

$$D=An^4+Bn^2+C=0, \tag{13}$$

where $n=ck/\omega$ is the refractive index. Normalizing again, as before, we rewrite (13) as

$$D=A\xi^4(1+\eta^2)^2+\beta\xi^2(1+\eta^2)\frac{\alpha^2}{\lambda}+C\frac{\alpha^4}{\lambda^2}=0, \tag{14}$$

where $\lambda=c^2/v_{\text{th}\perp e}$.

The coefficients A, B and C are given by (Freund, Dillenburg & Wu 1982)

$$A=\epsilon_{xx}\frac{1}{(1+\eta^2)}+\epsilon_{zz}\left(\frac{\eta^2}{\eta^2+1}\right)+(\epsilon_{xz}+\epsilon_{zx})\frac{\eta}{1+\eta^2}-\frac{i}{k}\left(\frac{\partial_y\epsilon_{yx}}{(1+\eta^2)^{\frac{1}{2}}}+\frac{1}{1+\eta^2}\partial_y\epsilon_{yz}\right), \tag{15}$$

$$\begin{aligned}B={}&-\epsilon_{xx}\epsilon_{zz}-(\epsilon_{yy}\epsilon_{zz}-\epsilon_{yz}\epsilon_{zy})\frac{\eta^2}{1+\eta^2}+\epsilon_{xz}\epsilon_{zx}\\&-(\epsilon_{xx}\epsilon_{yy}-\epsilon_{xy}\epsilon_{yx})\frac{1}{1+\eta^2}+\{\epsilon_{xy}\epsilon_{yz}+\epsilon_{yx}\epsilon_{zy}\\&-\epsilon_{yy}(\epsilon_{xz}+\epsilon_{zx})\}\frac{\eta}{1+\eta^2}+\frac{i}{kx}\left(\epsilon_{zz}+\epsilon_{yy}\frac{1}{(1+\eta^2)}\right.\\&\left.\times\,\partial_y\epsilon_{yx}+\left(\epsilon_{yy}\frac{\eta}{1+\eta^2}-\epsilon_{zx}\right)\partial_y\epsilon_{yz}-\left(\epsilon_{yx}\frac{1}{1+\eta^2}+\epsilon_{yz}\frac{\eta}{1+\eta^2}\right)\partial_y\epsilon_{yy}\right\},\end{aligned} \tag{16}$$

$$\begin{aligned}C={}&\epsilon_{zz}(\epsilon_{xx}\epsilon_{yy}-\epsilon_{xy}\epsilon_{yx})-\epsilon_{xx}\epsilon_{yz}\epsilon_{zy}\\&-\epsilon_{yy}\epsilon_{xz}\epsilon_{zx}-\epsilon_{yy}\epsilon_{xz}\epsilon_{zx}+\epsilon_{xy}\epsilon_{yz}\epsilon_{zx}\\&+\epsilon_{yx}\epsilon_{zy}\epsilon_{xz}+(i/k_x)\{(\epsilon_{yz}\epsilon_{zy}-\epsilon_{yy}\epsilon_{zz})\,\partial_y\epsilon_{yx}\\&+(\epsilon_{yy}\epsilon_{zx}-\epsilon_{yx}\epsilon_{zy})\,\partial_y\epsilon_{yz}+(\epsilon_{zz}\epsilon_{yx}-\epsilon_{yz}\epsilon_{zx})\,\partial_y\epsilon_{yy}\}.\end{aligned} \tag{17}$$

The general dispersion is now applied to the specific model under consideration and can be written as

$$D \simeq D_b + D_p = 0, \tag{18}$$

where D_b is the dispersion relation of the beam and D_p the dispersion relation of the plasma. Since $n_b \ll n_p$ and we consider the beam–plasma interaction around resonance, i.e. when $\alpha = \eta b^{\frac{1}{2}} \xi \; (\omega \approx k_z v_z)$ the dispersion relation of the beam makes no significant contribution to the real part of the general dispersion relation.

The growth rate is obtained under the assumption that $\alpha = \alpha_r + i\gamma$ where $|\gamma/\alpha_r| \ll 1$. Thus expansion of the dispersion function in powers of the growth rate $\gamma (\equiv \omega_i/\omega_{ce})$ is given by the following expression for our particular model:

$$\gamma = -\operatorname{Im} D_b(\mathbf{k}, \omega_r) \Big/ \left[\frac{\partial}{\partial \omega_r} D_p(\mathbf{k}, \omega_r)\right]. \tag{19}$$

Around resonance the argument of the Fried–Conte dispersion dispersion function for the beam is very small and thus expansion as a power series is possible (Fried & Conte 1961)

$$Z_b(x) = i\pi^{\frac{1}{2}} \exp(-x^2) - 2x(1 - \tfrac{2}{3}x^2 + \dots) \quad (x \ll 1). \tag{20}$$

At this point the arguments of the Fried–Conte dispersion function for the electron and ion components of the plasma are large, since the beam velocity is considered much larger than the parallel electron and ion thermal spreads. Thus we can write down an asymptotic expansion for the Fried–Conte dispersion function for the plasma. This expansion is of the form

$$Z_{i,e}(y) = i\pi^{\frac{1}{2}} \exp(-y^2) - \frac{1}{y}\left(1 + \frac{1}{2y^2} + \dots\right) (y \gg 1). \tag{21}$$

Using (20) and (21) we can obtain expressions for $\operatorname{Im} D_b$ and D_p.

$$\operatorname{Im} D_b = A_i \xi^4 (1+\eta^2)^2 + B_i \xi^2 (1+\eta^2)\, \alpha_r^2/\lambda + C_i \alpha_r^4/\lambda^2,$$

$$D_p = A_r \xi^4 (1+\eta^2)^2 + B_r \xi^2 (1+\eta^2)\, \alpha_r^2/\lambda + C_r \alpha_r^4/\lambda^2,$$

where the indices i and r correspond to the imaginary and real parts respectively.

From (11), (12), (20) and (21) we get

$$\begin{aligned}
A_i &= \frac{-4\pi^{\frac{1}{2}} a\theta}{\alpha_r^2} \frac{b}{d_1} \left(\frac{\alpha_r}{\eta d_1^{\frac{1}{2}} \xi} - \frac{\psi_b d_1^{\frac{1}{2}}}{2\eta} + \left(\frac{b}{d_1}\right)^{\frac{1}{2}}\right) \Lambda_b \frac{\eta^2}{(1+\eta^2)}, \\
B_i &= \frac{4\pi^{\frac{1}{2}} a\theta}{\alpha_r^2} \frac{b}{d_1} \left(\frac{\alpha_r}{\eta d_1^{\frac{1}{2}} \xi} - \frac{\psi_b d_1^{\frac{1}{2}}}{2\eta} + \left(\frac{b}{d_1}\right)^{\frac{1}{2}}\right) \Lambda_b \frac{(1+2\eta^2)}{(1+\eta^2)} + \frac{\pi^{\frac{1}{2}} a\theta\, d_2^2 \xi^2 \psi_b}{2\alpha_r^2 \eta d_1^{\frac{1}{2}}} F_b, \\
C_i &= \frac{-4\pi^{\frac{1}{2}} a\theta}{\alpha_r^2} \frac{b}{d_1} \left(\frac{\alpha_r}{\eta d_1^{\frac{1}{2}} \xi} - \frac{\psi_b d_1^{\frac{1}{2}}}{2\eta} + \frac{b}{d_1}\right) \Lambda_b - \frac{\pi^{\frac{1}{2}} a\theta\, d_2^2 \xi^2 \psi_b}{2\alpha_r^2 \eta d_1^{\frac{1}{2}}},\ F_b,
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
A_r = 1 &- \frac{2a\psi_p}{\alpha^2} \frac{\eta}{1+\eta^2} \left[\frac{g^{\frac{1}{2}}}{\sigma_e} F_e + \frac{(\beta h_1)^{\frac{1}{2}}}{\sigma_i} F_i\right] \\
&+ \cos^2\theta \left\{\frac{4a}{\alpha^2} \left[\left\{\left(2 + \frac{1}{2\sigma_e}\right) \frac{\psi_p g^{\frac{1}{2}}}{2\eta} + \sigma_e^2 (g+1) + \tfrac{1}{2}\right\} \Lambda_e + \tfrac{1}{2}\right]\right. \\
&\left. + \frac{4\alpha\beta}{\alpha^2} \left[\left\{\left(2 + \frac{1}{2\sigma_i}\right) \frac{\psi_p h_1^{\frac{1}{2}}}{2\eta \beta^{\frac{1}{2}}} + \sigma_i^2 \left(\frac{h_1}{h_2} + 1\right) + \tfrac{1}{2}\right\} \Lambda_i + \tfrac{1}{2}\right]\right\},
\end{aligned}$$

$$B_r = -\epsilon_{zz}^p - \epsilon_{yy}^p \frac{\eta^2}{1+\eta^2}(\epsilon_{zz}^p - 1) - \epsilon_{yy}^p + \frac{a^2\xi^2\cos^2\theta}{\alpha^2}\left[\frac{g^{\frac{1}{2}}}{\sigma_e}F_e + \frac{(\beta h_1)^{\frac{1}{2}}}{\sigma_i}F_i\right]$$
$$-\frac{6a^2\xi^2\psi_p}{\alpha^4}\frac{\eta}{1+\eta^2}\left[\left(1+\frac{1}{g}\right)F_e + \left(h_2 + \frac{h_2^2}{h_1}\right)F_i\right]\left[\frac{g^{\frac{1}{2}}}{\sigma_e}F_e + \frac{(\beta h_1)^{\frac{1}{2}}}{\sigma_i}F_i\right]$$
$$+\frac{2a\psi_p}{\alpha^2}\frac{\eta}{1+\eta^2}\left[\frac{g^{\frac{1}{2}}}{\sigma_e}F_e + \frac{(\beta h_1)^{\frac{1}{2}}}{\sigma_i}F_i\right]\left[1 - \frac{2a\xi^2}{\alpha^2}(\epsilon_{yy}^p - 1)\right]$$
$$-\frac{a\xi}{\alpha^2 k_x}\frac{\eta}{1+\eta^2}\left[\frac{g^{\frac{1}{2}}}{\sigma_e}F_e + \frac{(\beta h_1)^{\frac{1}{2}}}{\sigma_i}F_i\right]\left[1 - \frac{2a\xi\psi_p k_x}{\alpha^2}\left\{\left(1+\frac{1}{g}\right)F_e + (1+h_2)F_i\right\}\right]$$

$$C_r = \epsilon_{zz}^p - \frac{2a\xi^2}{\alpha^2}\left[1 + \frac{2a}{\alpha^2} + \frac{2a\beta}{\alpha^2}\right]\left[\left(1+\frac{1}{g}\right)F_e + (h_2+1)F_i\right]$$
$$-\frac{8a^2\xi^2}{\alpha^4}F_e\Lambda_e\left[\left\{\left(1+\frac{1}{g}\right)\frac{\psi_p g^{\frac{1}{2}}}{\eta} + \frac{\sigma_e^2(g+1)^2}{g} + \frac{1}{2}\left(1+\frac{1}{g}\right)\right\} + \frac{\psi\sigma_e(g+1)}{2\eta g^{\frac{1}{2}}}\right]$$
$$-\frac{8a^2\xi^2\beta}{\alpha^4}F_e\Lambda_i\left[\left\{\left(1+\frac{1}{g}\right)\frac{\psi_p h_1^{\frac{1}{2}}}{\eta\beta^{\frac{1}{2}}} + \sigma_i^2\left(1+\frac{1}{g}\right)\left(\frac{h_1}{h_2}+1\right) + \frac{1}{2}\left(1+\frac{1}{g}\right)\right\}\right.$$
$$\left. + \frac{\psi_p}{2\eta g^{\frac{1}{2}}}\frac{\sigma_i^2}{\sigma_e}\left(\frac{h_1}{h_2}+1\right)\right] - \frac{8a^2\xi^2}{\alpha^4}F_i\Lambda_e\left[\left\{(h_2+1)\frac{\psi_p g^{\frac{1}{2}}}{\eta} + (h_2+1)\sigma_e^2(g+1)\right.\right.$$
$$\left.\left. + \tfrac{1}{2}(h_2+1)\right\} + \frac{\psi_p h_2^2}{2\eta(\beta h_1)^{\frac{1}{2}}}\frac{\sigma_e^2}{\sigma_i}(g+1)\right] - \frac{8a^2\xi^2\beta}{\alpha^4}F_i\Lambda_i\left[\left\{(h_2+1)\frac{\psi_p h_1^{\frac{1}{2}}}{\eta\beta^{\frac{1}{2}}}\right.\right.$$
$$\left.\left. + \sigma_i^2(h_2+1)\left(\frac{h_1}{h_2}+1\right) + \tfrac{1}{2}(h_2+1)\right\} + \frac{\psi h_2^2}{2\eta(\beta h_1)^{\frac{1}{2}}}\sigma_i\left(\frac{h_1}{h_2}+1\right)\right], \tag{23}$$

where

$$\sigma_e = \alpha/\eta\xi g^{\frac{1}{2}}$$

and

$$\sigma_i = \alpha/\eta\xi(h_1\beta)^{\frac{1}{2}}.$$

In the local approximation it is necessary that the projection of the wavelengths of the excited modes in the direction perpendicular to both the externally applied magnetic field and the direction of the density gradient must be much larger than the electron Larmor radius, i.e. $\xi > 1$. Now it may be noted that the arguments of F and Λ are much larger than unity. Thus an asymptotic expansion for these quantities is possible.

From Luke (1975) we have

$$F(a;c;u) \sim \frac{\Gamma(c)\,e^u u^{a-c}}{\Gamma(a)}; \quad u \gg 1 \tag{24}$$

and from Ichimaru (1973) we have

$$\Lambda(v) = (2\pi v)^{-\frac{1}{2}}; \quad v \gg 1. \tag{25}$$

We can then obtain an expression for the growth rate γ (equation 19) with the help of (11), (12), (15)–(17), (20)–(25).

It may be noted here that in (21) and (22) terms proportional to λ^{-1} and λ^{-2} make small contributions unless thermal velocities or the beam bulk velocity is comparable to the velocity of light. Thus electromagnetic effects are in general small.

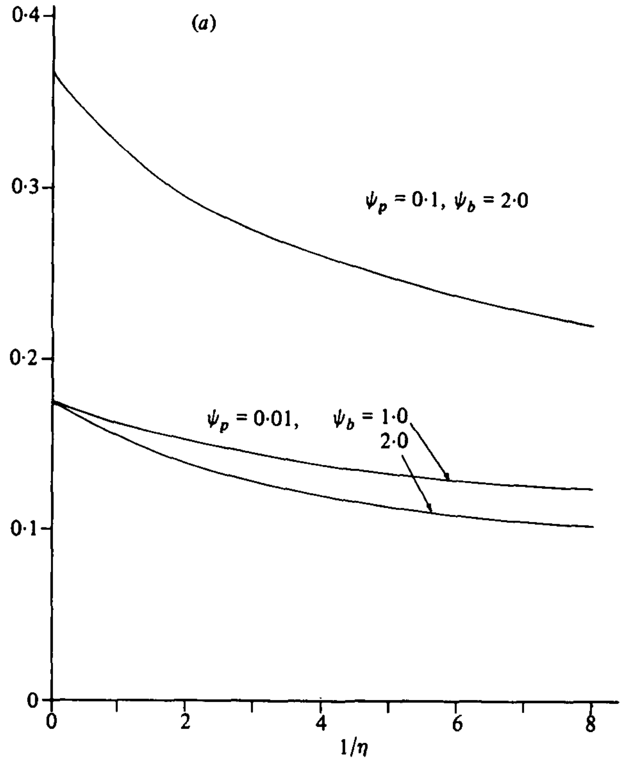

FIGURE 1. The plots of (*a*) frequency α_r and (*b*) growth rate γ of the most unstable wave as a function of $1/\eta$ ($=\tan\theta$ where θ is the angle of propagation with respect to $\boldsymbol{B}_0$) with ψ_p and ψ_b as parameters. Other plasma and beam parameters are

$$\omega_{pe}^2/\omega_{ce}^2 = 0{\cdot}1,\ \omega_b^2/\omega_{ce}^2 = 0{\cdot}01,\ V_z^2/V_{\mathrm{th}\perp e}^2 = 10{\cdot}0,\ T_{\parallel i}/T_{\perp e} = 1{\cdot}0,\ T_{\parallel e}/T_{\perp e} = T_{\perp i}/T_{\perp e} = 0{\cdot}5,\ T_{\parallel b}/T_{\perp e} = T_{\perp b}/T_{\perp e} = 5{\cdot}0,\ m_e/m_i = 1/1836.$$

We also note here that $\gamma < 0$ corresponds to growth since, in obtaining (1), plane wave solutions were assumed to have the form

$$\exp i(\omega t - \mathbf{k}.\mathbf{r});\quad \omega = \omega_r + i\omega_i.$$

Now we derive the expressions for the growth rate of the whistler mode instability in the electrostatic limit. The dispersion relation in this limit is obtained by putting $A = 0$ for both beam and plasma components in (13). Using (19) and expressions for A_i and A_r from (22) and (23) respectively, we obtain an expression for the growth rate:

$$\gamma = -\frac{\pi^{\frac12}\theta b\alpha_r\Lambda_b}{d_1}\left\{\frac{\alpha_r}{\eta\xi d_1^{\frac12}} - \frac{\psi_b d_1^{\frac12}}{2\eta} + \left(\frac{b}{d_1}\right)^{\frac12}\right\} \Bigg/ \Bigg[2\left\{\left[\frac{\xi g\psi_p}{2\alpha_r} - \frac{\psi_p g^{\frac12}}{\eta} - \frac{\alpha_r^2(g+1)}{\xi^2\eta^2 g} + \tfrac12\right]\Lambda_e + \tfrac12\right\} + 2\beta\left\{\left\{\frac{\psi_p h_1^{\frac12}}{\eta\beta^{\frac12}} + \frac{\xi h_1\psi_p}{2\alpha_r} - \frac{\alpha_r^2}{\xi^2\eta^2 h_1\beta}\left(\frac{h_1}{h_2}+1\right) + \tfrac12\right\}\Lambda_i + \tfrac12\right\} - \frac{3}{2}\frac{\psi_p\xi}{\alpha_r}(gF_e + \beta h_1 F_i)\Bigg]. \tag{26}$$

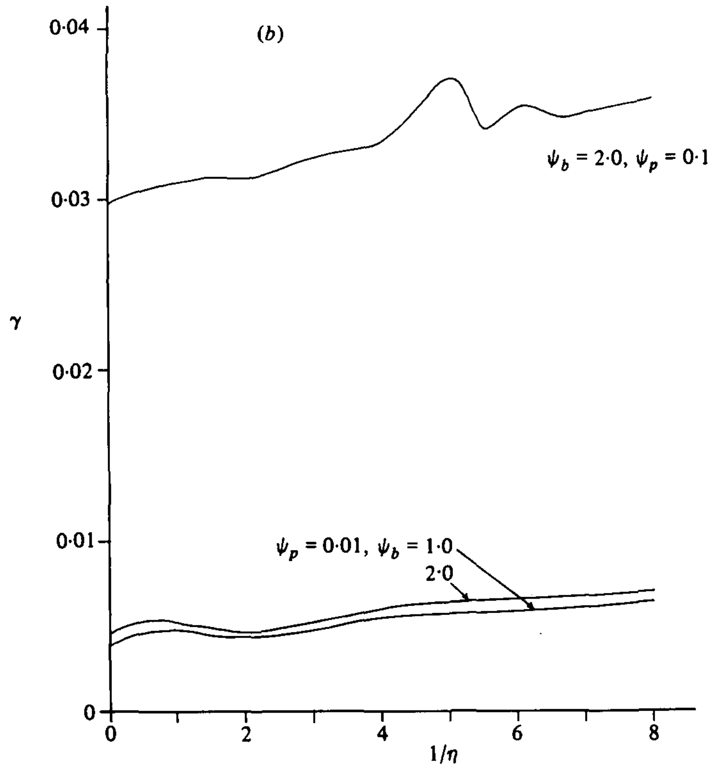

FIGURE 1(*b*). For legend see opposite.

Equation (26) shows that the growth rate depends on the inhomogeneities of both the beam and the plasma in a complicated fashion.

4. Numerical analysis and discussion

In this section we consider the whistler mode instability in the electrostatic limit and obtain solutions corresponding to the most unstable modes. Figures 1(*a*) and 1(*b*) show the most unstable frequencies and their growth rates as a function of the angle of propagation with respect is the ambient magnetic field $\mathbf{B}_0$ for different values of ψ_p and ψ_b. It is seen that the frequency and growth rate of the instability increase with the magnitude of ψ_p. However, the effect of increasing the beam inhomogeneity parameter ψ_b is to cause a decrease in the frequency of the instability especially at large angles. The growth rate increases with ψ_b as in the case of variation with ψ_p. It is also noted (not shown in figures 1(*a*) and (*b*)) that the effect of plasma and beam temperatures is mainly to reduce the growth rate of the instability.

To summarize, we have considered an inhomogeneous suprathermal beam of electrons which interacts with an inhomogeneous warm plasma. Gradients exist in the number densities of the beam and the plasma. A general dispersion relation including electromagnetic effects is derived for our system. In the electrostatic limit we have investigated the whistler mode excited as a result of Čerenkov

($m = 0$) interaction. It is found that the effect of the inhomogeneities is to increase the growth rate of the instabilities.

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