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Oblique propagation of nonlinear magnetosonic waves

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We consider obliquely propagating (with respect to the ambient field) nonlinear magnetosonic waves in a hot plasma with arbitrary beta. It is shown that the two-dimensional propagation of both the fast and the slow modes is governed by the Kadomstev–Petiashvilli soliton equation. Explicit expressions are obtained for the various physical quantities involved via the soliton solution.

We consider the propagation of two-dimensional nonlinear magnetosonic waves in a hot plasma. It is shown that the propagation of such waves may be governed by the Kadomstev–Petiashvilli (KP) equation (Kadomstev & Petiashvilli 1970), which is, in fact, the two-dimensional version of the Korteveg–de Vries (KdV) soliton equation and is valid for small angles of propagation. Recently De Vito & Pantano (1984) have investigated the propagation of two-dimensional magnetosonic waves using the KP equation for the case of a cold plasma. In this case, only one mode (intermediate) propagates, but in the case of a hot plasma, as will be shown in this work, both fast and slow modes appear, and have soliton solutions.

We consider a system of Cartesian co-ordinates, in which the ambient magnetic field B_0 lies in the (x,y) plane making a small angle θ with the x axis, and propagation is in the (x,z) plane. The basic equations that we use are the one-fluid isotropic isothermal magnetohydrodynamic equations with the generalized Ohm's law. These equations are (Boyd & Sanderson 1969)

$$\frac{\partial \rho'}{\partial t}+\nabla.(\rho'\mathbf{v}) = 0, \tag{1}$$

$$\rho'\frac{d\mathbf{v}}{dt} = -\nabla P'+\frac{1}{c}(\mathbf{v}\times\mathbf{B}'), \tag{2}$$

$$\mathbf{E}' = \frac{\mathbf{v}\times\mathbf{B}'}{c}+\frac{m^+}{\rho' ec}(\mathbf{j}'\times\mathbf{B}')+\frac{m^+}{2\rho' e}\nabla P', \tag{3}$$

$$\nabla\times\mathbf{B}' = \frac{4\pi}{c}\mathbf{j}', \tag{4}$$

$$\nabla\times\mathbf{E}' = -\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t}, \tag{5}$$

$$P'/\rho' = \text{const.}, \tag{6}$$

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where $d/dt = \partial/\partial t + \mathbf{v}.\nabla$; ρ', $\mathbf{v}$ and P' are the mass density, hydrodynamic velocity and pressure, respectively; m^+ is the ion mass; e is the electron charge; $\mathbf{E}'$ and $\mathbf{B}'$ are the electric and magnetic field vectors and $\mathbf{j}$ is the current density. Substituting (4) into (2), we get

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P + \frac{v_A^2}{\rho}(\nabla\times\mathbf{B})\times\mathbf{B} \tag{7}$$

and eliminating E' from (3) by using (5) yields

$$\frac{\partial B}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \frac{v_A^2}{\Omega_i}\nabla\times\frac{1}{\rho}[(\nabla\times\mathbf{B})\times\mathbf{B}]. \tag{8}$$

In (7) and (8) we use the dimensionless variables $\rho = \rho'/\rho_0$, $B = B'/B_0$ and $P' = P'/P_0$, where the subscript zero refers to the background quantities and v_A and Ω_i are the Alfvén velocity and the ion gyrofrequency.

To develop equations governing the propagation of magnetosonic waves we use the reductive perturbation scheme as given in the work of De Vito & Pantano (1984). By expanding the fluctuating variables in terms of a small quantity ϵ we have

$$\left.\begin{aligned}
\rho &= 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + \dots,\\
v_x &= \epsilon u_1 + \epsilon^2 u_2 + \dots,\\
v_y &= \epsilon v_1 + \epsilon^2 v_2 + \dots,\\
v_z &= \epsilon^{\frac{3}{2}} W_1 + \epsilon^{\frac{5}{2}} W_2 + \dots,\\
B_x &= B_0\cos\theta,\\
B_y &= B_0\sin\theta + \epsilon B_{y1} + \epsilon^2 B_{y2} + \dots,\\
B_z &= \epsilon^{\frac{3}{2}} B_{z1} + \epsilon^{\frac{5}{2}} B_{z2} + \dots .
\end{aligned}\right\} \tag{9}$$

We note here that all fluctuating quantities are functions of x, z and t. Quantities subscripted 1 and 2 are fluctuating quantities in order of smallness, respectively. We also introduce the stretched variables

$$\left.\begin{aligned}
\xi &= \epsilon^{\frac{1}{2}}(x - v_{\text{ph}}t),\\
\eta &= \epsilon z,\\
\tau &= \epsilon^{\frac{3}{2}} t,
\end{aligned}\right\} \tag{10}$$

where v_{ph} is a constant which is calculated below.

We now substitute (9) and (10) into (1), (7) and (8) and collect terms of lowest order in ϵ (order $\frac{3}{2}$):

$$\left.\begin{aligned}
-v_A\lambda\frac{\partial\rho_1}{\partial\xi} + \frac{\partial u_1}{\partial\xi} &= 0,\\
-\frac{\lambda}{v_A}\frac{\partial u_1}{\partial\xi} &= \beta\frac{\partial\rho_1}{\partial\xi} - \sin\theta\frac{\partial B_{y1}}{\partial\xi},\\
-\frac{\lambda}{v_A}\frac{\partial v_1}{\partial\xi} &= \cos\theta\frac{\partial B_{y1}}{\partial\xi},\\
-v_A\lambda\frac{\partial B_{y1}}{\partial\xi} &= -\sin\theta\frac{\partial u_1}{\partial\xi} + \cos\theta\frac{\partial v_1}{\partial\xi},
\end{aligned}\right\} \tag{11}$$

where $\beta = c_s^2/v_A^2$ and $c_s^2 = P_0/\rho_0$, and λ is the normalised phase velocity given by

$$\lambda = v_{\text{ph}}/v_A.$$

Putting $\rho_1 = \phi$, we can express the fluctuating quantities u_1, v_1 and B_{y1}, by using the set of equations (11). Thus we get

$$\begin{pmatrix} \rho_1 \\ u_1 \\ v_1 \\ B_{y1} \end{pmatrix} = \phi \begin{pmatrix} 1 \\ v_A \lambda \\ (\lambda^2 - \beta)/\sin\theta \\ \dfrac{-\cos\theta}{\sin\theta} \dfrac{v_A(\lambda^2-\beta)}{\lambda} \end{pmatrix} \tag{12}$$

From (11) and (12) we obtain

$$\lambda = \frac{1+\beta}{2}\left[1 \pm \left(1 - \frac{4\beta\cos\theta}{(1+\beta)^2}\right)^{\frac{1}{2}}\right]. \tag{13}$$

Expression (13) is the linear dispersion relation for magnetosonic waves in two dimensions (Alexandrov, Bogdankevich & Rukhadze 1984), written in a dimensionless form. The upper sign corresponds to the fast mode and the lower sign to the slow magnetosonic mode.

Now we develop the KP soliton equation and write a one-soliton solution for both the fast and slow modes. By substituting (9) and (10) into (1), (7) and (8) and collecting terms of order ϵ^2 and $\epsilon^{\frac{5}{2}}$, we have respectively

$$\left.\begin{aligned} -\frac{\lambda}{v_A}\frac{\partial W_1}{\partial \xi} &= \sin\theta\frac{\partial B_{y1}}{\partial \eta} + \cos\theta\frac{\partial}{\partial \xi}B_{z1} - \beta\frac{\partial}{\partial \eta}\rho_1, \\ -\lambda\frac{\partial B_{z1}}{\partial \xi} &= \frac{\cos\theta}{v_A}\frac{\partial W_1}{\partial \xi} - \frac{v_A}{\Omega_i}\cos\theta\frac{\partial^2 B_{y1}}{\partial \xi^2} \end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned} &\frac{\partial \rho_1}{\partial \tau} - \frac{\partial u_2}{\partial \xi} + \rho_1\frac{\partial u_1}{\partial \xi} + u_1\frac{\partial \rho_1}{\partial \xi} + \frac{\partial W_1}{\partial \eta} = \lambda v_A \frac{\partial \rho_2}{\partial \xi}, \\ &\frac{\partial u_1}{\partial \tau} + u_1\frac{\partial u_1}{\partial \xi} - v_A^2\beta\rho_1\frac{\partial \rho_1}{\partial \xi} + v_A^2 B_{y1}\frac{\partial B_{y1}}{\partial \xi} - v_A^2\rho_1\sin\theta\frac{\partial B_{y1}}{\partial \xi} \\ &\qquad = v_A\lambda\frac{\partial u_2}{\partial \xi} - v_A^2\beta\frac{\partial \rho_2}{\partial \xi} - v_A^2\sin\theta\frac{\partial B_{y2}}{\partial \xi}, \\ &\frac{\partial v_1}{\partial \tau} + u_1\frac{\partial v_1}{\partial \xi} + v_A^2\cos\theta\frac{\partial B_{y1}}{\partial \xi} = \lambda v_A\frac{\partial v_2}{\partial \xi} + v_A^2\cos\theta\frac{\partial B_{y2}}{\partial \xi}, \\ &\frac{\partial B_{y1}}{\partial \tau} + u_1\frac{\partial B_{y1}}{\partial \xi} + B_{y1}\frac{\partial u_1}{\partial \xi} = \lambda v_A\frac{\partial B_{y2}}{\partial \xi} - \sin\theta\frac{\partial W_1}{\partial \eta} \\ &\qquad + \frac{v_A^2\cos\theta}{\Omega_i}\frac{\partial^2 B_{z1}}{\partial \xi^2} - \sin\theta\frac{\partial u_2}{\partial \xi} + \cos\theta\frac{\partial v_2}{\partial \xi}. \end{aligned}\right\} \tag{15}$$

From the set of equations (14) and expression (12), we get the following two equations for B_{z1} and W_1 in terms of ϕ and λ.

$$\frac{\partial B_{z1}}{\partial \xi} = \frac{\lambda^2\cos\theta}{\cos^2\theta - \lambda^2}\frac{\partial \phi}{\partial \eta} - \frac{\lambda v_A(\lambda^2-\beta)}{\Omega_i(\cos^2\theta-\lambda^2)}\frac{\cos\theta}{\sin\theta}\frac{\partial^2\phi}{\partial \xi^2}, \tag{16}$$

$$\frac{\partial W_1}{\partial \xi} = \frac{-\lambda v_A}{\cos^2\theta-\lambda^2}\frac{\partial \phi}{\partial \eta} + \frac{v_A^2\cos^2\theta\,(\lambda^2-\beta)}{\Omega_i\sin\theta\,(\cos^2\theta-\lambda^2)}\frac{\partial^2\phi}{\partial \xi^2}. \tag{17}$$

We now eliminate quantities with subscripts 2 and terms containing B_{z1} and W_1 from (15), by using (12), (13), (16) and (17). Omitting the rather lengthy and cumbersome calculations, we obtain

$$\frac{\partial}{\partial \xi}\left[\frac{\partial \phi}{\partial \tau}+p\phi\frac{\partial \phi}{\partial \xi}-q\frac{\partial^3 \phi}{\partial \xi^3}\right]+r\frac{\partial^2 \phi}{\partial \eta^2} = 0, \tag{18}$$

where p, q, r are given by

$$p = \frac{v_A\lambda^3[3\lambda^4-\beta\{6\lambda^2-(3\beta+2\sin^2\theta)\}]}{\lambda^6+\lambda^4(1-2\beta)+\lambda^2\beta\{1-3\cos^2\theta+\beta\}+\beta^2\cos^2\theta}, \tag{19}$$

$$q = \frac{v_A^3\lambda^3(\lambda^2-\beta)^2\cos^2\theta}{\Omega_i^2(\lambda^2-\cos^2\theta)\,[\lambda^6+\lambda^4(1-2\beta)+\lambda^2\beta\{1-3\cos^2\theta+\beta\}+\beta^2\cos^2\theta]}, \tag{20}$$

$$r = \frac{v_A\lambda^7\sin^2\theta}{(\lambda^2-\cos^2\theta)\,[\lambda^6+\lambda^4(1-2\beta)+\lambda^2\beta\{1-3\cos^2\theta+\beta\}+\beta^2\cos^2\theta]}. \tag{21}$$

By making the transformations

$$\phi = -(12/pq)\,\phi', \quad \eta = \eta'/(12q/r)^{\frac{1}{2}}, \quad \tau = -\tau'/q, \tag{22}$$

equation (18) acquires the same form as Satsuma's (1976) equation (2) taken with the lower sign (which corresponds to propagation in a medium with positive dispersion). Thus (18) is the KP equation for the propagation of magnetosonic solitons in two dimensions. Following Satsuma (1976) we can write the solution to (18), namely

$$\phi = \phi_0 \operatorname{sech}^2\alpha \tag{23}$$

where $\phi_0 = -(12q/p)K_\xi^2$, $\alpha = K_\xi\xi+K_\eta\eta-\omega\tau$ and K_ξ, K_η and ω are related by the dispersion relation

$$-\omega K_\xi+4K_\xi^4-rK_\eta^2 = 0. \tag{24}$$

We have obtained the KP equation (18) for the propagation of two-dimensional magnetosonic solitons in a medium with positive dispersion. In the expressions for p, q and r, the upper sign corresponds to fast magnetosonic solitons and the lower sign to the case of slow magnetosonic solitons. The magnetosonic soliton propagates with a phase velocity which is obtained from equation (24) and is given by

$$\frac{\omega}{K} = \frac{\omega}{(K_\xi^2+K_\eta^2)^{\frac{1}{2}}} = \frac{4K_\xi^3 q}{(K_\xi^2+K_\eta^2)^{\frac{1}{2}}}-\frac{rK_\eta^2}{K_\xi(K_\xi^2+K_\eta^2)^{\frac{1}{2}}} \tag{25}$$

making an angle θ with the x axis. It should be noted here that this angle is expected to be small because the effect of the perpendicular direction is small (Kadomstev & Petiashvilli 1970). However, it has been pointed out by Satsuma (1976) that from a mathematical point of view such a restriction does not hold. For small angles of propagation, the fast magnetosonic soliton propagates as a negative pulse and the slow soliton as a positive pulse. This situation may, however, reverse for large angles of propagation beyond some critical angle θ_c.

We are now able to calculate expressions for B_{z1} and W_1 from (14) and (23).

These are given by

$$B_{z1} = \frac{\lambda^2 \cos\theta}{\cos^2\theta - \lambda^2} \frac{K_\eta}{K_\xi} \phi + \frac{2\lambda v_A(\lambda^2 - \beta)}{\Omega_i(\cos^2\theta - \lambda^2)} \frac{\cos\theta}{\sin\theta} K_\xi \phi \tanh\alpha, \tag{25}$$

$$W_1 = \frac{-\lambda^3 v_A}{\cos^2\theta - \lambda^2} \frac{K_\eta}{K_\xi} \phi - \frac{2v_A^2(\lambda^2 - \beta)\cos^2\theta}{\Omega_i(\cos^2\theta - \lambda^2)\sin\theta} K_\xi \phi \tanh\alpha. \tag{26}$$

By substituting (2) into (3) and using (12), (13), (23), (25) and (26), we find expressions for the electric field components of the solitary magnetosonic wave. To lowest order in ϵ (i.e. $\frac{3}{2}$), we obtain for the x and y components, respectively,

$$\frac{cE_x}{B_0} = \frac{-\lambda^3 v_A \sin^2\theta}{\cos\theta(\cos^2\theta - \lambda^2)} \phi - \frac{2v_A^2 \cos^3\theta(\lambda^2 - \beta)}{\Omega_i(\cos^2\theta - \lambda^2)} K\phi \tanh\alpha$$
$$+ \frac{3v_A^2\beta}{\Omega_i} K \cos\theta\phi \tanh\alpha + \frac{2\lambda^2 v_A^2}{\Omega_i} K \cos\theta\phi \tanh\alpha, \tag{27}$$

$$\frac{cE_y}{B_0} = \frac{\lambda^3 v_A \sin^2\theta}{(\cos^2\theta - \lambda^2)\cos\theta} \phi + \frac{2v_A^2 \cos^3\theta}{\Omega_i(\cos^2\theta - \lambda^2)} (\lambda^2 - \beta)\phi \tanh\alpha$$
$$+ \frac{v_A^2 \cos^2\theta}{\Omega_i \sin\theta} K(\lambda^2 - \beta)\phi \tanh\alpha. \tag{28}$$

For E_z we need to compute to order ϵ, and we obtain

$$\frac{cE_z}{B_0} = -v_A \lambda \sin\theta \left[1 + \frac{\cos^2\theta}{\sin^2\theta}\left(\frac{\lambda^2 - \beta}{\lambda^2}\right)\right]\phi. \tag{29}$$

From the above expressions we see that the electric field disappears as α approaches ∞.

We note here that, for the case of parallel propagation ($\theta = 0$), equation (18) reduces to a standard KdV equation, as in the case of ion acoustic waves investigated by Washimi & Taniuti (1966).

To sum up, we have shown that nonlinear magnetosonic waves are governed by the Kadomstev–Petiashvilli equation which exhibits soliton solutions. Since we have considered propagation in a hot plasma with no restrictions imposed upon β, both fast and slow mode solitons appear. This is different from the cold plasma case investigated by De Vito & Pantano (1984) where only one mode (intermediate) appeared. The presence of β also complicates the dependence of the soliton solution, and the quantities associated with it, on the propagation angle. From the soliton solutions, we were also able to obtain expressions (25), (26) and (27) which give the electric fields related to both fast and slow magnetosonic solitons.

Such waves may find application in various astrophysical and laboratory plasma situations. For example in the review by Porkolab & Chang (1978) it was pointed out that the shocks are accompanied by intense ordered oscillations (solitons) in laboratory situations as well as simulation studies.

More recently, Formisano (1985), whilst studying magnetosonic astrophysical shocks, pointed out that these are often accompanied by large-amplitude magnetosonic waves. Tsurutani & Smith (1984) have identified large-amplitude obliquely propagating magnetosonic waves. However, it may be noted here that conclusive proof for the existence of any type of soliton in astrophysical and space plasmas is not available. One possible reason for this may be that data analysts have tried to look for unique relationships between velocity and amplitude of the large-amplitude waves (Hoppe *et al.* 1981). However, such relationships exist only for one-dimensional propagation.

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