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Cusp and Regular Ion-Acoustic Solitons

M. N. S. Qureshi · Jiankui Shi · H. A. Shah

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Abstract Cups and regular smooth solitons are studied using the fluid model in cylindrical geometry for parallel propagating ion-acoustic waves in a low β plasma. It is found that smooth solitons only occur in the supersonic regime, whereas cusp solitons occur both in supersonic and subsonic regimes. In the supersonic regime, the amplitude and the width of cusp solitons increase when the Mach number M increases and initial electric field E_0 decreases. However, the amplitude of smooth regular solitons increases and their width decreases when E_0 increases and M decreases. For the subsonic case, both the amplitude as well as the width of a cusp solitons increase when M increases and E_0 decreases. Corresponding to these cusp and regular solitons, bipolar electric field structures are also studied. These results may be helpful in understanding the properties of ion-acoustic regular and cusp solitons in space plasmas.

Keywords Cusp soliton · Ion-acoustic waves

1 Introduction

Using the ideal fluid equations, ion-acoustic and ion-cyclotron waves in collisionless plasmas have been studied extensively [1–8]. Results have shown that ion-acoustic and ion-cyclotron waves can evolve into nonlinear density periodic waves and density solitons under different plasma conditions. In addition to the normal smooth solitons, solitons with cusp profiles have also been found in plasmas. Observations on ion-acoustic density fluctuations [9–12] and density spiky pulses [13] have been reported by Freja satellite in the auroral upper ionospheric region. These nonlinear electrostatic waves are responsible for the acceleration of upflowing ions [14, 15] and electron acceleration [16] in the auroral region. Theoretical studies on cusp solitons have also been carried out by a number of researchers, e.g., Porkolab and Goldman [17] reported cusp solitons while studying the upper hybrid waves and two stream instabilities with warm two fluid equations; Shapiro [18] studied the modulational interaction of the lower hybrid waves with a kinetic Alfven mode and found cusp solitons. Yinhua et al. [19] studied the nonlinear dust kinetic Alfven waves perpendicular to the external magnetic field by employing the pseudopotential formulism and reported smooth regular solitons as well as cusp solitons in the electron density corresponding to the singularities in the Sagdeev potential profiles in the subsonic regimes. Wei and Chen [20] also investigated the presence of cusp solitons along with the regular smooth solitons perpendicular to the external magnetic field with pseudopotential technique during their study of nonlinear lower hybrid waves in two-ion-species plasma.

In almost all the theoretical studies on cusp solitons [1–7, 17–20], solitary solutions have been found for either oblique or perpendicular (with respect to the background magnetic field) propagation only. So far to the best of our knowledge, no theoretical study has been reported on ion-acoustic cusp solitons propagating parallel to the magnetic field. In this paper, we present an ion fluid model in a low plasma β [ratio of gas pressure to magnetic pressure ($8\pi\ p/B^2$)], magnetized plasma with cylindrical symmetry and derive the "Sagdeev potential" for propagation parallel to the external magnetic

M. N. S. Qureshi (✉) · H. A. Shah
Department of Physics, GC University,
Lahore 54000, Pakistan
e-mail: nouman_sarwar@yahoo.com

M. N. S. Qureshi · J. Shi
State Key Laboratory of Space Weather, CSSAR, Chinese Academy of Sciences,
Beijing 100190, China

field. We note here that the Sagdeev potential technique has been the most successful approach in the study of nonlinear structures, such as solitons since the last century. In his pioneering work, Sagdeev [21] used an analogy to an oscillator, i.e., motion of a point mass trapped in a potential well. Sagdeev [21] noted that the nonlinear differential equation describing the evolution of waves could be cast in the same form as the equation describing the motion of a particle in a potential well, thus this approach has come to be known as the as the Sagdeev potential method, and as pointed out above, this has been used extensively in the investigation of solitary waves and other nonlinear structures. Linear analysis of the model fluid equations shows that the ion-acoustic wave could develop into a nonlinear wave. It is found that not only cusp solitons but also smooth solitons, both in subsonic and supersonic regimes, can exist in low β plasmas.

2 Physical Formulation

The fluid consists of electrons and ions, in a $\beta<<1$ plasma. The magnetic field is directed along the z-axis; we are merely looking for electrostatic solutions and the magnetic field will be passively taken into account in the gyro-frequency. Using a cylindrical coordinate system and neglecting the electron inertia, the fluid equations for the ions in cylindrical coordinates can be written as

$$\frac{\partial n}{\partial t}+\frac{\partial(nv_r)}{\partial r}+\frac{\partial(nv_z)}{\partial z}=-\frac{nv_r}{r} \tag{1}$$

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}=-\frac{1}{m_i n}\frac{\partial p}{\partial r}-\frac{e}{m_i}\frac{\partial \varphi}{\partial r}+\frac{v_\theta^2}{r}+\Omega_i v_\theta \tag{2}$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}=-\frac{v_r v_\theta}{r}-\Omega_i v_r \tag{3}$$

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{m_i n}\frac{\partial p}{\partial z}-\frac{e}{m_i}\frac{\partial \varphi}{\partial z} \tag{4}$$

with

$$p=nT_i \tag{5}$$

$$n=n_i\approx n_e\approx n_0\exp(e\varphi/T_e) \tag{6}$$

where p is the thermal pressure, e is the elementary charge, φ is the electric potential, v is the ion velocity, and n_0 is constant number density. First, we linearize the set of Eqs. 1–4 and obtain the linear dispersion relation as $\omega^2\left(\omega^2-aC_s^2k_z^2\right)\left(\omega^2-\Omega_i^2\right)=0$. Here, ω is the wave frequency, the coefficient $a=T_i/T_e+1$, $C_s=(T_e/m_i)^{1/2}$is the ion-acoustic velocity, k_z is wave vector in the z-direction, and $\Omega_i=eB_0/(m_ic)$is the ion-cyclotron frequency. From dispersion relation, we obtain the following solutions, $\omega^2=a\,C_s^2k_z^2$ and $\omega^2=\Omega_i^2$, which show an ion-acoustic wave can be excited in a plasma and consequently may develop into nonlinear waves.

3 Nonlinear Analysis

We rewrite Eqs. 1–4 by shifting to co-moving frame using the variable $\xi=(k_r\,r+k_z\,z-\omega t)\cdot\Omega_i/\omega=(\alpha\,R+\gamma\,Z-M\,\tau)/M$ and the following dimensionless quantities are introduced: $N=n/n_0, \tau=\Omega_i t, R=r/\rho_i, Z=z/\rho_i, V=v/C_s, \Phi=e\varphi/T_e, M=v_p/C_s, v_p=\omega/k$, to obtain

$$-\frac{dN}{d\xi}+\frac{\alpha}{M}\frac{d(NV_R)}{d\xi}+\frac{\gamma}{M}\frac{d(NV_Z)}{d\xi}=-\frac{NV_R}{R} \tag{7}$$

$$-\frac{dV_R}{d\xi}+\frac{\alpha\,V_R}{M}\frac{dV_R}{d\xi}+\frac{\gamma\,V_Z}{M}\frac{dV_R}{d\xi}=-\frac{a\,\alpha}{M}\frac{d(\ln N)}{d\xi}+\frac{V_\theta^2}{R}+V_\theta \tag{8}$$

$$-\frac{dV_\theta}{d\xi}+\frac{\alpha V_R}{M}\frac{dV_\theta}{d\xi}+\frac{\gamma\,V_Z}{M}\frac{dV_\theta}{d\xi}=-\frac{V_RV_\theta}{R}-V_R \tag{9}$$

$$-\frac{dV_Z}{d\xi}+\frac{\alpha\,V_R}{M}\frac{dV_Z}{d\xi}+\frac{\gamma\,V_Z}{M}\frac{dV_Z}{d\xi}=-\frac{a\,\gamma}{M}\frac{d(\ln N)}{d\xi} \tag{10}$$

where N=exp(Φ), α=sinθ, γ=cosθ, and k_r and k_z are the components of K in the directions of r and z, respectively. Now, we consider the wave propagation along the magnetic field, i.e., $K\|B$ so that, α=0 and γ=1 and following standard algebraic techniques using the boundary conditions when $V|_{\xi=0}=0$, $N|_{\xi=0}=1$, we obtain [22]

$$\frac{1}{2}\left(\frac{dN}{d\xi}\right)^2+\Psi(N)=0 \tag{11}$$

where

$$\Psi(N)=\frac{\left[\left(N\sqrt{1-\frac{2a}{M^2}\ln N}-1\right)^2-\left(\frac{a}{M^2}-1\right)^2E_0^2\right]\left(N\sqrt{1-\frac{2a}{M^2}\ln N}\right)^2}{2\left[\frac{a}{M^2}-\left(1-\frac{2a}{M^2}\ln N\right)\right]^2} \tag{12}$$

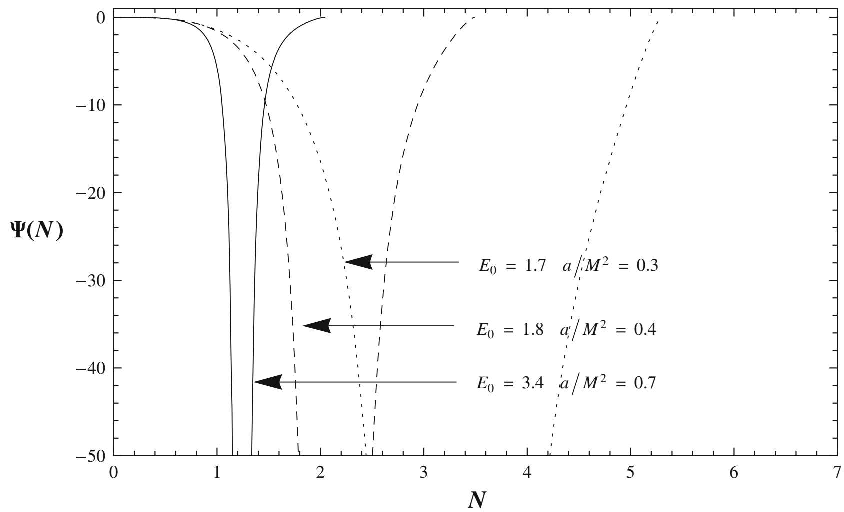

Fig. 1 Profile of Sagdeev potentials "$\Psi(N)$" for different values of E_0 and M when a/M^2 <1

Here, E_0 is the initial value of E. Equation 11 is the energy integral of a classical particle in a 1-D "potential well" and $\Psi(N)$ in Eq. 12 is so-called the Sagdeev potential. The nonlinear solution from Eq. 12 can be obtained numerically when the "Sagdeev potential" $\Psi(N)$ <0. Equation 12 has solutions when the plasma parameters satisfy either of the conditions below

$$\left|\left(a/M^2-1\right)E_o\right|>1,\quad a/M^2<1 \tag{13}$$

$$\left|\left(a/M^2-1\right)E_o\right|>1,\quad a/M^2>1,\quad G_m\leq\ 1+\left|\left(a/M^2-1\right)E_o\right| \tag{14}$$

where $G_m=\sqrt{a/M^2}\exp\left[\left(1-a/M^2\right)/\left(2a/M^2\right)\right]$ and $\Psi(N)$ has properties: $\Psi(0)\rightarrow 0$, $\partial\Psi(0)/\partial N=0$.

3.1 Nonlinear Solution When a/M^2 <1

Figure 1 is plotted for the different values of a/M^2 and E_0 for the condition 13. Here, the ion-to-electron temperature ration is taken as 10^{-1}, so the parameter a= 1.1 throughout this paper. We note that there exists infinite depth Sagdeev potential with $\Psi(N_1)\ \rightarrow\ -\infty$ at N_1 where $1<N_1<N_{\max}$. Ideally, a particle would start slowly from N=1, attains infinite speed at N=N_1, and will pass the infinite well with infinite speed and then slow down until it reaches the opposite wall. The particle then returns to the point N=1 in a symmetrical way. The corresponding solitary wave would then be a cusp like solitons but with a smooth top [23]. At two points on its shoulders, the Sagdeev potential has infinite derivatives but all the physical quantities remain continuous. On the other hand, in the infinite well, the

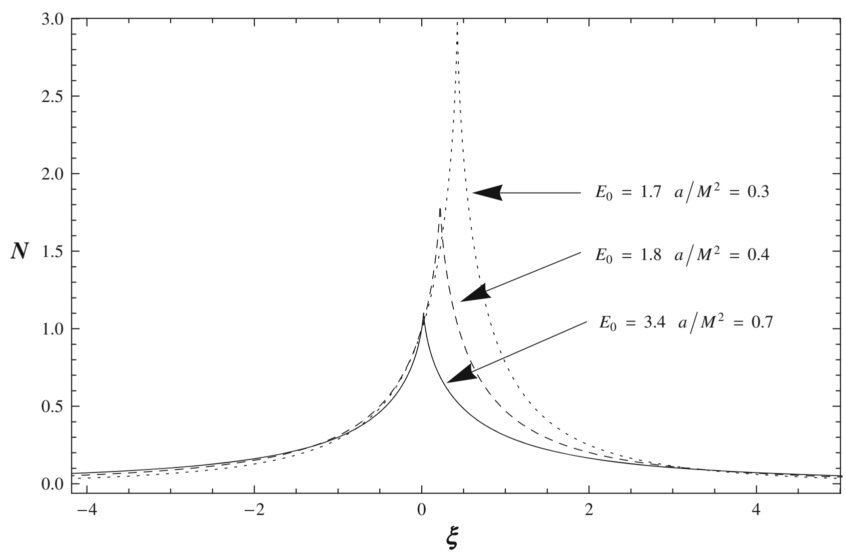

Fig. 2 Cusp soliton structures for the same values of E_0 and M corresponding to the Sagdeev potential "$\Psi(N)$" in Fig. 1

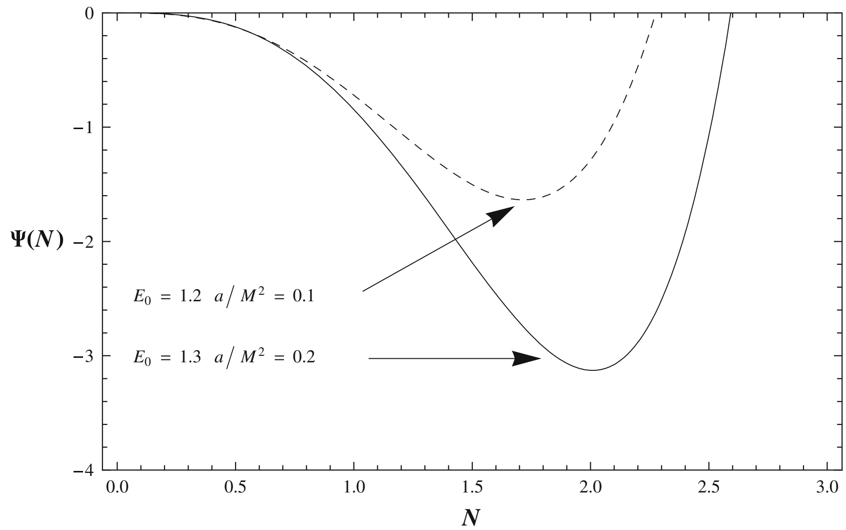

Fig. 3 Profile of usual smooth Sagdeev potentials "$\Psi(N)$" for the different values of E_0 and M when a/M^2 <1

particle motion is extremely sensitive to any deviation from the ideal condition. If we assume that it is reflected at point (N_1), we will get a corresponding cusped density hump solitons [17, 20]. This type of solitons is quite different from the regular smooth solitons corresponding to the Sagdeev potential in Fig. 3 and shall be investigated here. Figure 2 depicts the cusped density hump solitons corresponding to the Sagdeev potential in Fig. 1 for the same parameters. We can see that the amplitude of the cusped solitons increases while the width decreases when E_0 decreases and Mach number M increases.

Figure 3 is plotted for the different values of a/M^2 and E_0 for the condition 13. It is to be noted that regular Sagdeev potential profile can also exist under the same condition 13 when $\Psi(N)$ has properties: $\Psi(0)\rightarrow 0$, $\partial\Psi(0)/\partial N = 0$, $\Psi(N_2) = 0$, and $\partial\Psi(N_2)/\partial N > 0$ where $N_2 = \exp(M^2/2a) > 1$. The analogous particle in such a well will pass through the well and reaches the opposite wall and get reflected there. The corresponding density hump soliton would then be a regular smooth soliton. Such smooth solitons are shown in Fig. 4 for the same parameters given in Fig. 3. We can see that the width decreases slightly and amplitude increases for the smooth solitons when E_0 increases and Mach number M decreases.

3.2 Nonlinear Solution When a/M^2 >1

Figure 5 is plotted for the different values of a/M^2 and E_0 when the condition 14 is satisfied. Again, we can note that there exist an infinite depth Sagdeev potential with $\Psi(N_1) \rightarrow -\infty$ at N_1 and at two points on its shoulders the Sagdeev potential has infinite derivatives. On the basis of the same analysis done in the last subsection, we will get a corresponding cusped hump solitons. Cusped density hump solitons corresponding to the Sagdeev potential shown in

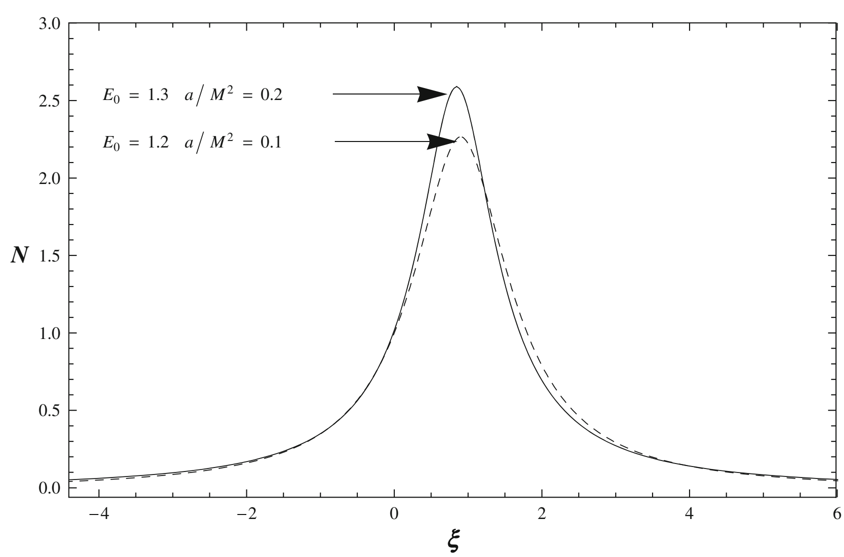

Fig. 4 Regular smooth solitons for the same values of E_0 and M corresponding to the Sagdeev potential "$\Psi(N)$" in Fig. 3

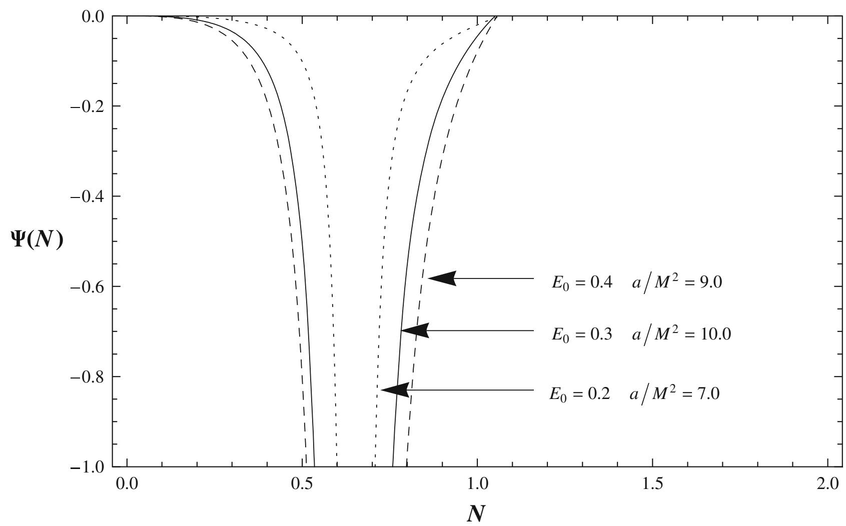

Fig. 5 Profile of Sagdeev potentials "$\Psi(N)$" for the different values of E_0 and M when $a/M^2 > 1$

Fig. 5 are depicted in Fig. 6 for the same parameters. We can note that both width and amplitude of the cusped solitons increase when E_0 decreases and Mach number M increases similar to the case when $a/M^2 < 1$.

3.3 Bipolar Electric Field Solitary Structures

By definition, we can obtain the electric field as $E = -\partial_\xi \Phi = -(1/N)\left(\partial_\xi N\right)$. Therefore, corresponding to the regular smooth solitons, we can obtain the bipolar electric field solitary structures as shown in Fig. 7. We can see that the amplitude of bipolar EFSs increases when E_0 increases and Mach number M decreases. We can also obtain the electric field profiles corresponding to the cusp solitons, which are consisting of two adjacent spikes, one positive and one negative. Such an electric field wave form is shown in a schematic way in Fig. 8.

4 Discussion and Conclusion

Solitons with cusp profiles occur because of a balance of the dispersive and nonlinear effects close to the wave breaking point and have been found in both fluids and plasmas [17–19]. In our theoretical model, we have assumed that the phase velocity lies between ion and electron thermal velocities so that the Landau damping can be neglected. However, in real situations, Landau damping always exists and prevent the Sagdeev potential from becoming infinitely deep. Moreover, the cusps where the gradients are very large are expected to be smoothed out somewhat by high-order nonlinearities or dissipative process such as those arising from shear or turbulent viscosity [19, 20].

In this paper, we consider an ion fluid model for the solitary waves propagating along the magnetic field lines,

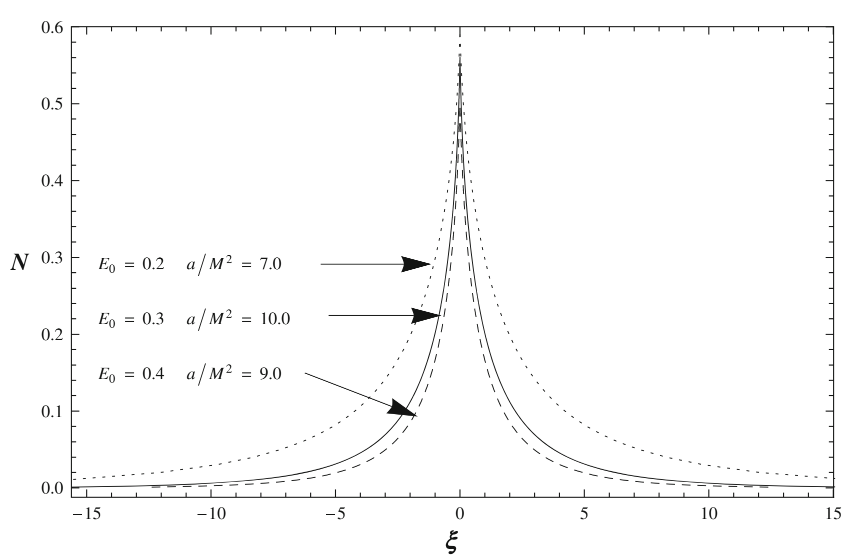

Fig. 6 Cusp solitons for the same values of E_0 and a/M^2 corresponding to the Sagdeev potential "$\Psi(N)$" in Fig. 5

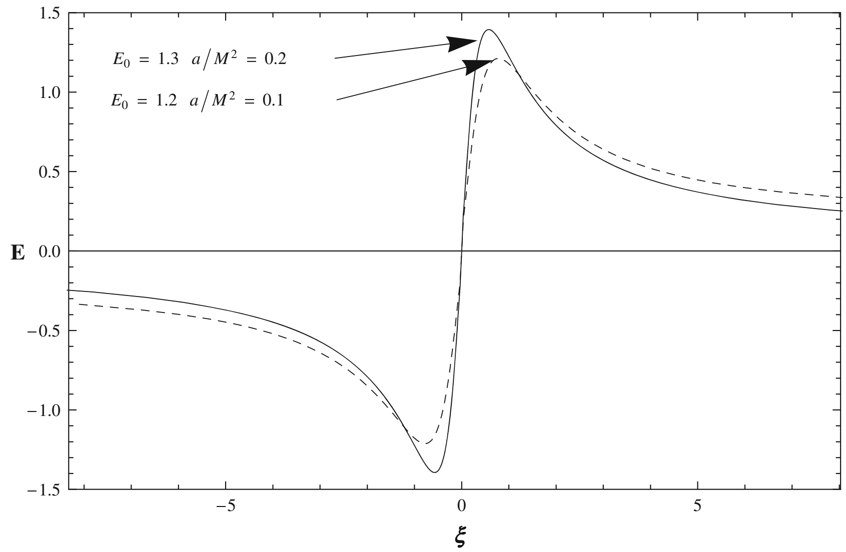

Fig. 7 Electric field structures corresponding to regular smooth solitons

applicable to space plasmas in particular the regions where plasma β is much smaller than unity. Data from Fast and Polar satellites [24, 25] from the auroral zone at altitudes between 5,000 and 10,000 km show that plasma has a small beta. We further note that in most space plasmas, as well as in the plasma of the auroral region, collisions are negligible (i.e., the collision frequency is much smaller than the characteristic plasma frequency). If the electrons are hot, then thermal effects dominate over the magnetic effects and electrons can be considered to be Boltzmannian to a very good approximation. This is in accord with the observations mentioned above [9–12, 24, 25].

The results from our model show that the cusp solitons and smooth solitons coexist in the supersonic regime ($a/M^2 < 1$), whereas in the subsonic regime ($a/M^2 > 1$), only cusp solitons exist. Figures 2 and 6 show the profiles of these solitons in the supersonic regime under condition 13 and in the subsonic regimes under condition 14, respectively. For the supersonic solitary waves, the amplitude and width of a cusp soliton increase when the Mach number M increases and E_0 decreases. However, the amplitude of smooth solitons increases and their width decreases when E_0 increases and M decreases. For the case of subsonic solitary waves, both the amplitude and the width of a hump soliton increase when M increases and E_0 decreases. Comparison of Figs. 2 and 6 shows that the amplitudes of supersonic cusp solitons are larger but the widths are smaller than that of subsonic cusp solitons. These density cusp solitons are similar to features found in observations: events of spiky density pulses have been reported by Freja spacecraft in the auroral region of Earth's ionosphere [9–12]. Corresponding to these cusp and regular solitons, bipolar electric field structures are also

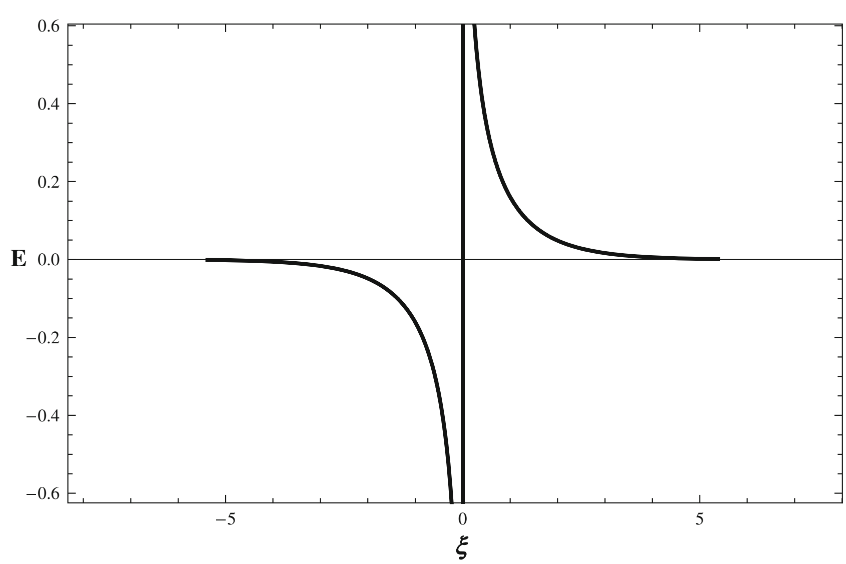

Fig. 8 Electric field structure corresponding to the cusp solitons

studied. Therefore, the above results may be helpful in understanding the properties of ion-cyclotron and ion-acoustic solitons in the space plasmas.

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