Punjab University Journal of Mathematics (2023),55(3),117-133
https://doi.org/10.52280/pujm.2023.550303

On the partition dimension of circulant graph $C_n(1, 2, 3, 4)$

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Received: 1 March,2022 / Accepted: 27 February, 2023 / Published online: 25 March, 2023

Abstract. Let $\Lambda = \{B_1, B_2, \ldots, B_l\}$ be an ordered l-partition of a connected graph $G(V(G), E(G))$. The partition representation of vertex x with respect to Λ is the l-vector, $r(x|\Lambda) = (d(x, B_1), d(x, B_2), \ldots, d(x, B_l))$, where $d(x, B) = min\{d(x, y)|y \in B\}$ is the distance between x and B. If the l - vectors $r(x|\Lambda)$, for all $x \in V(G)$ are distinct then l - partition is called a resolving partition. The least value of l for which there is a resolving l - partition is known as the partition dimension of G symbolized as $pd(G)$. In this paper, the partition dimension of circulant graphs $C_n(1, 2, 3, 4)$ is computed for $n \geq 8$ as,

$$pd(C_n(1,2,3,4)) = \begin{cases} n, & \text{if } 8 \leq n \leq 9; \\ 6, & \text{if } n = 10; \\ 5, & \text{if } n \geq 11. \end{cases}$$

AMS (MOS) Subject Classification Codes: 05C07
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Key Words: Circulant graphs, metric dimension, partition dimension.

1. Introduction and Preliminaries

Slater et al. [20] and Melter et al. [8] independently introduced the concept of metric dimension of a graph in 1975 and 1976 which has many applications in robotics [12], chemistry [2] and optimization [19]. Later Chartrand et al. [3] presented the notion of partition dimension a modified form of metric dimension. The computing the metric dimension is NP-hard [4], the problems become even harder when it comes to partition dimension where we have to find a resolving partition which contains sets instead of vertices. Further details of metric and partition dimension can be seen in the articles [1, 9, 15, 16, 17].

Let W be a connected graph with the vertex set $V(W)$ and edge set $E(W)$. For $u, v \in V(W)$, $d(u, v)$ denotes the length of shortest path between u and v. The distance between a vertex t and a set P is given as $d(t, P) = min\{d(t, x)|x \in P\}$. The diameter of W, symbolized by $diam(W)$, is the greatest distance between any two vertices. Let $\Omega = \{x_1, x_2, \dots, x_l\}$ be an ordered set of vertices, the representation of a vertex t with respect to Ω is the l - vector $r(t|\Omega) = (d(t, x_1), d(t, x_2), \dots, d(t, x_l))$. If the l - vectors $r(v|\Omega)$, for all $v \in V(W)$ are distinct then Ω is called a resolving set. The minimal value of l for which there is a resolving set is known as the metric dimension of G symbolized as $dim(G)$.

Let $\Lambda = \{B_1, B_2, \dots, B_l\}$ be an ordered l - partition of W. The partition representation of vertex v with respect to Λ is the l - vector $r(v|\Lambda) = (d(v, B_1), d(v, B_2), \dots, d(v, B_l))$. If the l - vectors $r(v|\Lambda)$, for all $v \in V$ are distinct then l - partition is called a resolving partition. The minimum l for which there is a resolving l - partition is called the partition dimension of W. The study of metric and partition dimension of different graphs has been an active area of research for the last two decades. Chartrand et al. [3] gave the comparison between the metric dimension and partition dimension and they also categorized the graphs having partition dimension 2 or n. The subsequent results from [3] have significant importance in our work.

Proposition 1.1. *If W is a connected graph of order $n \geq 2$ then*

(1) $pd(W) \leq \dim(W) + 1$;
(2) *W is path if and only if $pd(W) = 2$;*
(3) *W is the complete graph if and only if $pd(W) = n$.*

2. Circulant graphs

In the current section, we are interested in the special class of circulant graph $C_n(1,2,\ldots,t)$ containing vertices $v_0, v_1, \ldots, v_{n-1}$ with connection set {1, 2, . . . , t} for $1 \le t \le \lfloor n/2 \rfloor$. The distance between two vertices v_i and v_j in $C_n(1,2,\ldots,t)$, where $0 \le i < j < n$, is defined in [13] as follows:

$$d(v_i, v_j) = \begin{cases} \lceil \frac{j-i}{t} \rceil, & \text{if } 0 \le j-i \le \frac{n}{2}; \\ \lceil \frac{n-(j-i)}{t} \rceil, & \text{if } \frac{n}{2} < j-i < n \end{cases}$$

Many authors have computed the metric and partition dimension of different classes of circulant graphs [5, 6, 7, 10, 11, 13, 14, 18]. Imran et al. [10] discussed the metric dimension of circulant graphs $C_n(1,2,5)$. Salman et al. [18] discussed the metric and partition dimension of circulant graphs $C_n(1,2)$ and proved that partition dimension of circulant graph is 4 for $n \ge 6$ which was disproved by Grigorious et. al in [7]. Later in [14] Nadeem et al. corrected the partition dimension of $C_n(1,2)$ for $n \equiv 2(mod 4), n \ge 18$. Javaid et al. [11] studied the partition of circulant graphs $C_n(1,3)$ and $C_n(1,4)$. The subsequent proposition is given in [6].

Proposition 2.1. [6]
Consider the circulant graphs $C_n(1,2,\ldots,t)$ with $1 < t < \lfloor \frac{n}{2} \rfloor, n \ge (t+k)(t+1)$ and $n \equiv k \mod 2t$, then

(1) $pd(C_n(1,2,\ldots,t)) = t+1$, *when t is even and $gcd(k,2t) = 1$;*
(2) $pd(C_n(1,2,\ldots,t)) = t+1$, *when t is odd and $k = 2m, 1 \le m \le t-1$.*

Elizabeth et al. [13] disproved the claims in Proposition 2.1 with counterexamples and also gave the exact values of $pd(C_n(1,2,3))$. We summarize their results in Proposition 2.2 and 2.3.

Proposition 2.2. [13]
$pd(C_n(1,2,\ldots,t)) \le \frac{t}{2} + 4$, *whenever $n = 2lt$ for even $t \ge 4$ and $l \ge 2$.*

Proposition 2.3. [13]

$$pd(C_n(1,2,3)) = \begin{cases} n, & \text{if } 6 \le n \le 7; \\ 5, & \text{if } 8 \le n \le 9; \\ 4, & \text{if } n \ge 10. \end{cases}$$

The subsequent corollary is an easy consequence of Proposition 2.2.

Corollary 2.1. *For $l \ge 2$, $pd(C_n(1,2,3,4)) \le 6$ for $n = 8l$.*

In this paper, we generalize Corollary 2.1 and obtain the precise value of the partition dimension of $C_n(1,2,3,4)$.

3. Main Results

Throughout in the remaining part of the paper, we will denote $C_n(1,2,3,4)$ by G_n. It is clear from Proposition 1.1 that $pd(G_n) = n$ for $8 \leq n \leq 9$, because it is a complete graph. The diameter of G_n has been recently discussed in [5], we state this result in the following Proposition 3.1.

Proposition 3.1. [5] *If we write the order of G_n, as $n = 8k+r$ where $r \in \{2,3,\ldots,9\}$ then the diameter of G_n is $k+1$ and there are $r-1$ number of vertices at the diameter distance from any vertex v.*

The upper bound on $pd(G_n)$ for $n \geq 11$ is given in the subsequent theorem.

Theorem 3.1. *$pd(G_n) \leq 5$, for $n \geq 11$.*

Proof. The proof has eight subcases and a resolving partition, $\Lambda = \{A_1, A_2, A_3, A_4, A_5\}$ of $V(G_n)$ is given for each case. For our convenience we take $v_0 = v_n$.

Case 1: Let $n = 8k+2$. If $k \geq 2$, then consider $A_1 = \{v_i | 1 \leq i \leq 8k-9\}$,
$A_2 = \{v_{8k-8}, v_{8k-7}, v_{8k}\}$, $A_3 = \{v_{8k-6}, v_{8k-2}, v_{8k-1}\}$,
$A_4 = \{v_{8k-5}, v_{8k-4}, v_{8k-3}, v_{8k+2}\}$ and $A_5 = \{v_{8k+1}\}$. The Table 1, shows that Λ is resolving partition.

Table 1. $r(v|\Lambda)$ for $n = 8k+2$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4\delta+1} (0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+1$	$\delta+1$	$\delta+1$
$v_{4\delta+2} (0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+2$	$\delta+1$	$\delta+1$
$v_{4\delta+3} (0 \leq \delta \leq \alpha-2)$	0	$\delta+2$	$\delta+2$	$\delta+1$	$\delta+1$
$v_{4\delta+4} (0 \leq \delta \leq \alpha-3)$	0	$\delta+2$	$\delta+2$	$\delta+1$	$\delta+2$
$v_{4\alpha-4}$	0	$\alpha-1$	α	$\alpha-1$	α
$v_{4\alpha-3}$	0	$\alpha-1$	α	α	α
$v_{4\alpha-2}$	0	$\alpha-1$	$\alpha-1$	α	α
$v_{4\alpha-1}$	0	$\alpha-1$	$\alpha-1$	$\alpha-1$	α
$v_{8\alpha-4\delta-3} (2 \leq \delta \leq \alpha-1)$	0	$\delta-1$	δ	δ	$\delta+1$
$v_{8\alpha-4\delta-2} (2 \leq \delta \leq \alpha-1)$	0	$\delta-1$	$\delta-1$	δ	$\delta+1$
$v_{8\alpha-4\delta-1} (2 \leq \delta \leq \alpha-1)$	0	$\delta-1$	$\delta-1$	$\delta-1$	$\delta+1$
$v_{8\alpha-4\delta} (3 \leq \delta \leq \alpha)$	0	$\delta-2$	$\delta-1$	$\delta-1$	$\delta+1$
$v_{8\alpha-8}$	1	0	1	1	3
$v_{8\alpha-7}$	1	0	1	1	2
$v_{8\alpha}$	1	0	1	1	1
$v_{8\alpha-6}$	1	1	0	1	2
$v_{8\alpha-2}$	2	1	0	1	1
$v_{8\alpha-1}$	1	1	0	1	1
$v_{8\alpha-5}$	1	1	1	0	2
$v_{8\alpha-4}$	2	1	1	0	2
$v_{8\alpha-3}$	2	1	1	0	1
$v_{8\alpha+2}$	1	1	1	0	1
$v_{8\alpha+1}$	1	1	1	1	0

Case 2: Let $n = 8\alpha + 3$. If $\alpha = 1$, then consider $A_1 = \{v_1, v_2\}$, $A_2 = \{v_3, v_4, v_5, v_6, v_7, v_9\}$, $A_3 = \{v_8\}, A_4 = \{v_{10}\}$ and $A_5 = \{v_{11}\}$.

It can be verified easily that Λ is a resolving partition.

If $\alpha \geq 2$, then consider $A_1 = \{v_i | 1 \leq i \leq 8\alpha - 7\} \cup \{v_{8\alpha-5}, v_{8\alpha-3}, v_{8\alpha-1}\}$,

$A_2 = \{v_{8\alpha-6}, v_{8\alpha-2}, v_{8\alpha}, v_{8\alpha+2}\}$, $A_3 = \{v_{8\alpha-4}\}, A_4 = \{v_{8\alpha+1}\}$ and

$A_5 = \{v_{8\alpha+3}\}$. The Table 2, shows that Λ is resolving partition.

Table 2. $r(v|\Lambda)$ for $n = 8\alpha + 3$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4\delta+1}(0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+2$	$\delta+1$	$\delta+1$
$v_{4\delta+2}(0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+3$	$\delta+1$	$\delta+1$
$v_{4\delta+3}(0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+3$	$\delta+2$	$\delta+1$
$v_{4\delta+4}(0 \leq \delta \leq \alpha-3)$	0	$\delta+2$	$\delta+3$	$\delta+2$	$\delta+1$
$v_{4\alpha-4}$	0	α	α	α	$\alpha-1$
$v_{4\alpha-3}$	0	α	α	α	α
$v_{4\alpha-2}$	0	$\alpha-1$	α	α	α
$v_{4\alpha-1}$	0	$\alpha-1$	α	$\alpha+1$	α
$v_{4\alpha}$	0	$\alpha-1$	$\alpha-1$	$\alpha+1$	α
$v_{8\alpha-4\delta-3}(1 \leq \delta \leq \alpha-1)$	0	δ	δ	$\delta+1$	$\delta+2$
$v_{8\alpha-4\delta-2}(2 \leq \delta \leq \alpha-1)$	0	$\delta-1$	δ	$\delta+1$	$\delta+2$
$v_{8\alpha-4\delta-1}(2 \leq \delta \leq \alpha-1)$	0	$\delta-1$	δ	$\delta+1$	$\delta+1$
$v_{8\alpha-4\delta}(2 \leq \delta \leq \alpha-1)$	0	$\delta-1$	$\delta-1$	$\delta+1$	$\delta+1$
$v_{8\alpha-5}$	0	1	1	2	2
$v_{8\alpha-3}$	0	1	1	1	2
$v_{8\alpha-1}$	0	1	1	1	1
$v_{8\alpha-6}$	1	0	1	2	3
$v_{8\alpha-2}$	1	0	1	1	2
$v_{8\alpha}$	1	0	1	1	1
$v_{8\alpha+2}$	1	0	2	1	1
$v_{8\alpha-4}$	1	1	0	2	2
$v_{8\alpha+1}$	1	1	2	0	1
$v_{8\alpha+3}$	1	1	2	1	0

Case 3: Let $n = 8\alpha+4$. If $\alpha = 1$, then consider $A_1 = \{v_1, v_2, v_3, v_5, v_6, v_{11}\}$, $A_2 = \{v_4\}$, $A_3 = \{v_7\}, A_4 = \{v_8, v_9\}, A_5 = \{v_{10}, v_{12}\}$. It can be verified easily that Λ is a resolving partition.

If $\alpha \geq 2$, then consider $A_1 = \{v_i | 1 \leq i \leq 8\alpha - 6\} \cup \{v_{8\alpha-4}, v_{8\alpha-1}, v_{8\alpha}\}$, $A_2 = \{v_{8\alpha-5}, v_{8\alpha-3}\}, A_3 = \{v_{8\alpha-2}\}, A_4 = \{v_{8\alpha+1}, v_{8\alpha+2}, v_{8\alpha+3}\}$ and

$A_5 = \{v_{8\alpha+4}\}$. The Table 3, shows that Λ is resolving partition.

TABLE 3. $r(v|\Lambda)$ for $n = 8\alpha + 4$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4\delta+1}(0 \le \delta \le \alpha - 2)$	0	$\delta + 2$	$\delta + 2$	$\delta + 1$	$\delta + 1$
$v_{4\delta+2}(0 \le \delta \le \alpha - 2)$	0	$\delta + 3$	$\delta + 2$	$\delta + 1$	$\delta + 1$
$v_{4\delta+3}(0 \le \delta \le \alpha - 3)$	0	$\delta + 3$	$\delta + 3$	$\delta + 1$	$\delta + 1$
$v_{4\delta+4}(0 \le \delta \le \alpha - 3)$	0	$\delta + 3$	$\delta + 3$	$\delta + 2$	$\delta + 1$
$v_{4\alpha-5}$	0	α	$\alpha + 1$	$\alpha - 1$	$\alpha - 1$
$v_{4\alpha-4}$	0	α	$\alpha + 1$	α	$\alpha - 1$
$v_{4\alpha-3}$	0	α	$\alpha + 1$	α	α
$v_{4\alpha-2}$	0	α	α	α	α
$v_{4\alpha-1}$	0	$\alpha - 1$	α	α	α
$v_{4\alpha}$	0	$\alpha - 1$	α	$\alpha + 1$	α
$v_{8\alpha-4\delta-3}(1 \le \delta \le \alpha - 1)$	0	δ	$\delta + 1$	$\delta + 1$	$\delta + 2$
$v_{8\alpha-4\delta-2}(1 \le \delta \le \alpha - 1)$	0	δ	δ	$\delta + 1$	$\delta + 2$
$v_{8\alpha-4\delta-1}(2 \le \delta \le \alpha - 1)$	0	$\delta - 1$	δ	$\delta + 1$	$\delta + 2$
$v_{8\alpha-4\delta}(2 \le \delta \le \alpha - 1)$	0	$\delta - 1$	δ	$\delta + 1$	$\delta + 1$
$v_{8\alpha-4}$	0	1	1	2	2
$v_{8\alpha-1}$	0	1	1	1	2
$v_{8\alpha}$	0	1	1	1	1
$v_{8\alpha-5}$	1	0	1	2	3
$v_{8\alpha-3}$	1	0	1	1	2
$v_{8\alpha-2}$	1	1	0	1	2
$v_{8\alpha+1}$	1	1	1	0	1
$v_{8\alpha+2}$	1	2	1	0	1
$v_{8\alpha+3}$	1	2	2	0	1
$v_{8\alpha+4}$	1	2	2	1	0

Case 4: Let $n = 8\alpha + 5$. If $\alpha = 1$, then consider $A_1 = \{v_1\}$, $A_2 = \{v_2, v_3\}$,

$A_3 = \{v_4, v_9, v_{10}, v_{13}\}, A_4 = \{v_5, v_6, v_7, v_8, v_{12}\}$ and $A_5 = \{v_{11}\}$. It can be verified easily that Λ is a resolving partition.

If $\alpha \ge 2$, then consider $A_1 = \{v_i | 1 \le i \le 8\alpha - 5\} \cup \{v_{8\alpha}, v_{8\alpha+2}\}$,

$A_2 = \{v_{8\alpha-4}, v_{8\alpha-3}, v_{8\alpha+1}, v_{8\alpha+3}, v_{8\alpha+4}\}$, $A_3 = \{v_{8\alpha-2}\}, A_4 = \{v_{8\alpha-1}\}$ and $A_5 = \{v_{8\alpha+5}\}$. The Table 4, shows that Λ is resolving partition.

Case 5: Let $n = 8\alpha+6$. If $\alpha = 1$, then consider $A_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9\}$, $A_2 = \{v_7, v_8\}$, $A_3 = \{v_{10}, v_{13}\}, A_4 = \{v_{11}\}$ and

TABLE 4. $r(v|\Lambda)$ for $n = 8\alpha + 5$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4l+1}(0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+2$	$\delta+2$	$\delta+1$
$v_{4\delta+2}(0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+3$	$\delta+2$	$\delta+1$
$v_{4\delta+3}(0 \leq \delta \leq \alpha-2)$	0	$\delta+1$	$\delta+3$	$\delta+3$	$\delta+1$
$v_{4\delta+4}(0 \leq \delta \leq \alpha-3)$	0	$\delta+2$	$\delta+3$	$\delta+3$	$\delta+1$
$v_{4\alpha-4}$	0	α	$\alpha+1$	$\alpha+1$	$\alpha-1$
$v_{4\alpha-3}$	0	α	$\alpha+1$	$\alpha+1$	α
$v_{4\alpha-2}$	0	α	α	$\alpha+1$	α
$v_{4\alpha-1}$	0	α	α	α	α
$v_{4\alpha}$	0	$\alpha-1$	α	α	α
$v_{8\alpha-4\delta-3}(1 \leq \delta \leq \alpha-1)$	0	δ	$\delta+1$	$\delta+1$	$\delta+2$
$v_{8\alpha-4\delta-2}(1 \leq \delta \leq \alpha-1)$	0	δ	δ	$\delta+1$	$\delta+2$
$v_{8\alpha-4\delta-1}(1 \leq \delta \leq \alpha-1)$	0	δ	δ	δ	$\delta+2$
$v_{8\alpha-4\delta}(2 \leq \delta \leq \alpha-1)$	0	$\delta-1$	δ	δ	$\delta+2$
$v_{8\alpha}$	0	1	1	1	2
$v_{8\alpha+2}$	0	1	1	1	1
$v_{8\alpha-4}$	1	0	1	1	3
$v_{8\alpha-3}$	1	0	1	1	2
$v_{8\alpha+1}$	1	0	1	1	1
$v_{8\alpha+3}$	1	0	2	1	1
$v_{8\alpha+4}$	1	0	2	2	1
$v_{8\alpha-2}$	1	1	0	1	2
$v_{8\alpha-1}$	1	1	1	0	2
$v_{8\alpha+5}$	1	1	2	2	0

$A_5 = \{v_{12}, v_{14}\}$. It can be verified easily that Λ is a resolving partition.

If $\alpha \geq 2$, then consider $A_1 = \{v_i | 1 \leq i \leq 8\alpha-4\} \cup \{v_{8\alpha+6}\}$, $A_2 = \{v_{8\alpha-3}, v_{8\alpha-2}\}$, $A_3 = \{v_{8\alpha-1}, v_{8\alpha}, v_{8\alpha+1}, v_{8\alpha+3}\}$, $A_4 = \{v_{8\alpha+2}, v_{8\alpha+5}\}$ and $A_5 = \{v_{8\alpha+4}\}$. The Table 5, shows that Λ is resolving partition.

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Case 6: Let $n = 8\alpha+7$. If $\alpha = 1$, then consider $A_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{13}\}$,

$A_2 = \{v_7, v_8, v_{11}\}, A_3 = \{v_{10}, v_{14}\}$, $A_4 = \{v_{12}\}$ and $A_5 = \{v_{15}\}$. It can be verified easily that Λ is a resolving partition.

If $\alpha \geq 2$, then consider $A_1 = \{v_i | 1 \leq i \leq 8\alpha-3\}$,

$A_2 = \{v_{8\alpha-2}, v_{8\alpha-1}, v_{8\alpha+1}, v_{8\alpha+2}\}$,

$A_3 = \{v_{8\alpha}, v_{8\alpha+4}\}, A_4 = \{v_{8\alpha+3}, v_{8\alpha+6}, v_{8\alpha+7}\}$ and $A_5 = \{v_{8\alpha+5}\}$.

The Table 6, shows that Λ is resolving partition.

Table 5. $r(v|\Lambda)$ for $n = 8\alpha + 6$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4\delta+1}(0 \leq \delta \leq \alpha-2)$	0	$\delta+3$	$\delta+1$	$\delta+1$	$\delta+1$
$v_{4\delta+2}(0 \leq \delta \leq \alpha-2)$	0	$\delta+3$	$\delta+2$	$\delta+1$	$\delta+1$
$v_{4\delta+3}(0 \leq \delta \leq \alpha-2)$	0	$\delta+3$	$\delta+2$	$\delta+1$	$\delta+2$
$v_{4\delta+4}(0 \leq \delta \leq \alpha-2)$	0	$\delta+3$	$\delta+2$	$\delta+2$	$\delta+2$
$v_{4\alpha-3}$	0	α	α	α	α
$v_{4\alpha-2}$	0	α	$\alpha+1$	α	α
$v_{4\alpha-1}$	0	α	α	α	$\alpha+1$
$v_{8\alpha-4\delta-3}(1 \leq \delta \leq \alpha-1)$	0	δ	$\delta+1$	$\delta+2$	$\delta+2$
$v_{8\alpha-4\delta-2}(1 \leq \delta \leq \alpha-1)$	0	δ	$\delta+1$	$\delta+1$	$\delta+2$
$v_{8\alpha-4\delta-1}(1 \leq \delta \leq \alpha-1)$	0	δ	δ	$\delta+1$	$\delta+2$
$v_{8\alpha-4\delta}(1 \leq \delta \leq \alpha)$	0	δ	δ	$\delta+1$	$\delta+1$
$v_{8\alpha+6}$	0	2	1	1	1
$v_{8\alpha-3}$	1	0	1	2	2
$v_{8\alpha-2}$	1	0	1	1	2
$v_{8\alpha-1}$	1	1	0	1	2
$v_{8\alpha}$	1	1	0	1	1
$v_{8\alpha+1}$	2	1	0	1	1
$v_{8\alpha+3}$	1	2	0	1	1
$v_{8\alpha+2}$	1	1	1	0	1
$v_{8\alpha+5}$	1	2	1	0	1
$v_{8\alpha+4}$	1	2	1	1	0

Case 7: Let $n = 8\alpha+8$. If $\alpha = 1$ then consider $A_1 = \{v_1, v_2, v_3, v_4, v_5, v_7, v_{10}\}$,

$A_2 = \{v_6, v_8, v_9\}$, $A_3 = \{v_{11}, v_{12}, v_{14}\}, A_4 = \{v_{13}, v_{15}\}$ and $A_5 = \{v_{16}\}$.

It can be verified easily that Λ is a resolving partition.

If $\alpha \geq 2$, then consider $A_1 = \{v_i | 1 \leq i \leq 8\alpha - 3\} \cup \{v_{8\alpha+2}\}$,

$A_2 = \{v_{8\alpha-2}, v_{8\alpha-1}, v_{8\alpha}, v_{8\alpha+1}\}$, $A_3 = \{v_{8\alpha+3}, v_{8\alpha+6}\}, A_4 = \{v_{8\alpha+4}, v_{8\alpha+7}\}$ and $A_5 = \{v_{8\alpha+5}, v_{8\alpha+8}\}$. The Table 7, shows that Λ is resolving partition.

Case 8: Let $n = 8\alpha + 9$. If $\alpha \geq 1$, then consider $A_1 = \{v_i | 1 \leq i \leq 8\alpha - 2\}$,

$A_2 = \{v_{8\alpha-1}, v_{8\alpha}, v_{8\alpha+1}, v_{8\alpha+3}\}$, $A_3 = \{v_{8\alpha+2}, v_{8\alpha+8}\}$,
$A_4 = \{v_{8\alpha+4}, v_{8\alpha+7}, v_{8\alpha+9}\}$ and $A_5 = \{v_{8\alpha+5}, v_{8\alpha+6}\}$.
The Table 8, shows that Λ is resolving partition.

In all the above cases the partition representations are distinct, which completes the proof.

□

TABLE 6. $r(v|\Lambda)$ for $n = 8\alpha + 7$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4\delta+1}(0 \le \delta \le \alpha-1)$	0	$\delta+2$	$\delta+1$	$\delta+1$	$\delta+1$
$v_{4\delta+2}(0 \le \delta \le \alpha-2)$	0	$\delta+2$	$\delta+2$	$\delta+1$	$\delta+1$
$v_{4\delta+3}(0 \le \delta \le \alpha-2)$	0	$\delta+2$	$\delta+2$	$\delta+1$	$\delta+2$
$v_{4\delta+4}(0 \le \delta \le \alpha-2)$	0	$\delta+3$	$\delta+2$	$\delta+1$	$\delta+2$
$v_{4\alpha-2}$	0	α	$\alpha+1$	α	α
$v_{4\alpha-1}$	0	α	$\alpha+1$	α	$\alpha+1$
$v_{4\alpha}$	0	α	α	α	$\alpha+1$
$v_{8\alpha-4\delta-3}(0 \le \delta \le \alpha-1)$	0	$\delta+1$	$\delta+1$	$\delta+2$	$\delta+2$
$v_{8\alpha-4\delta-2}(1 \le \delta \le \alpha-1)$	0	δ	$\delta+1$	$\delta+2$	$\delta+2$
$v_{8\alpha-4\delta-1}(1 \le \delta \le \alpha-1)$	0	δ	$\delta+1$	$\delta+1$	$\delta+2$
$v_{8\alpha-4\delta}(1 \le \delta \le \alpha-1)$	0	δ	δ	$\delta+1$	$\delta+2$
$v_{8\alpha-2}$	1	0	1	2	2
$v_{8\alpha-1}$	1	0	1	1	2
$v_{8\alpha+1}$	1	0	1	1	1
$v_{8\alpha+2}$	2	0	1	1	1
$v_{8\alpha}$	1	1	0	1	2
$v_{8\alpha+4}$	1	1	0	1	1
$v_{8\alpha+3}$	2	1	1	0	1
$v_{8\alpha+6}$	1	1	1	0	1
$v_{8\alpha+7}$	1	2	1	0	1
$v_{8\alpha+5}$	1	1	1	1	0

THEOREM 3.2. *$pd(G_n) \ge 5$ for $n \ge 10$.*

Proof. We will show that $pd(G_n) \ne 4$ for $n \ge 10$

Assume that $pd(G_n) = 4$. Let $\Lambda = \{A_1, A_2, A_3, A_4\}$ be a resolving partition of $V(G_n)$. Clearly one of the sets A_1, A_2, A_3, A_4 contains at least 3 vertices so assume that $|A_1| \ge 3$. It is clear that there exist one vertex $v_i \in A_1$ such that $d(v_i, A_j) > 1$ for some $j \in \{2,3,4\}$ otherwise $r(v|\Lambda) = (0,1,1,1)$ for all $v \in A_1$. Without loss of generality consider $d(v_i, A_3) \ge 2$. Let v_j be a vertex in A_3 where $j > i$, s.t $d(v_i, v_j) = d(v_i, A_3)$. Let $V^* = \{v_{j-1}, v_{j-2}, v_{j-3}, v_{j-4}\}$ then no vertex in V^* belongs to A_3 as $d(v, v_i) < d(v_j, v_i)$ for all $v \in V^*$ also $d(v, A_3) = 1$ for all $v \in V^*$. Without loss of generality assume that $V^* \cap A_1 \ne \phi$.

Case 1: If all the elements of V^* are in A_1. i.e. $|V^* \cap A_1| = 4$ then $r(v_{j-4}|\Lambda) = (0, a, 1, a^{'}), r(v_{j-3}|\Lambda) = (0, b, 1, b^{'}), r(v_{j-2}|\Lambda) = (0, c, 1, c^{'})$, $r(v_{j-1}|\Lambda) = (0, d, 1, d^{'})$. Since $k+1$ is the diameter so $1 \le a, b, c, d, a^{'}$, $b^{'}, c^{'}, d^{'} \le k+1$.

Case 1.1 : If $k \le a, a^{'} \le k+1$.

The possible choices for $d(v, A_2)$ and $d(v, A_4)$ for $v \in V^*$ are shown in Tables 9 to 11. It is easy to that for $r = 2$ (see Table 9) and $r \ge 4$ (see Table 11) at least two representations will

TABLE 7. $r(v|\Lambda)$ for $n = 8\alpha + 8$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4\delta+1}(0 \leq \delta \leq \alpha - 1)$	0	$\delta + 2$	$\delta + 1$	$\delta + 1$	$\delta + 1$
$v_{4\delta+2}(0 \leq \delta \leq \alpha - 2)$	0	$\delta + 3$	$\delta + 1$	$\delta + 1$	$\delta + 1$
$v_{4\delta+3}(0 \leq \delta \leq \alpha - 2)$	0	$\delta + 3$	$\delta + 2$	$\delta + 1$	$\delta + 1$
$v_{4\delta+4}(0 \leq \delta \leq \alpha - 2)$	0	$\delta + 3$	$\delta + 2$	$\delta + 2$	$\delta + 1$
$v_{4\alpha-2}$	0	α	α	α	α
$v_{4\alpha-1}$	0	α	$\alpha + 1$	α	α
$v_{4\alpha}$	0	α	$\alpha + 1$	$\alpha + 1$	α
$v_{8\alpha-4\delta-3}(0 \leq \delta \leq \alpha - 1)$	0	$\delta + 1$	$\delta + 2$	$\delta + 2$	$\delta + 2$
$v_{8\alpha-4\delta-2}(1 \leq \delta \leq \alpha - 1)$	0	δ	$\delta + 2$	$\delta + 2$	$\delta + 2$
$v_{8\alpha-4\delta-1}(1 \leq \delta \leq \alpha - 1)$	0	δ	$\delta + 1$	$\delta + 2$	$\delta + 2$
$v_{8\alpha-4l}(1 \leq \delta \leq \alpha - 1)$	0	δ	$\delta + 1$	$\delta + 1$	$\delta + 2$
$v_{8\alpha-2}$	1	0	2	2	2
$v_{8\alpha-1}$	1	0	1	2	2
$v_{8\alpha}$	1	0	1	1	2
$v_{8\alpha+1}$	1	0	1	1	1
$v_{8\alpha+3}$	1	1	0	1	1
$v_{8\alpha+6}$	1	2	0	1	1
$v_{8\alpha+4}$	1	1	1	0	1
$v_{8\alpha+7}$	1	2	1	0	1
$v_{8\alpha+5}$	1	1	1	1	0
$v_{8\alpha+8}$	1	2	1	1	0

be same, leading to a contradiction. For $r = 3$, there are two vertices at $k+1$ distance so the representation $r(v|\Lambda) \neq r(w|\Lambda)$ for $v, w \in V^*$ if we either choose 2^{nd} or 3^{rd} column of Table 10 for $d(v, A_2)$ or $d(v, A_4)$.

Since we have $v_j \in A_3$ and $v_j, v_{j-1}, v_{j-2}, v_{j-3}, v_{j-4}$ are consecutive vertices with the connection set $\{1, 2, 3, 4\}$ so $r(v_j|\Lambda) = (1, k, 0, k)$.

Assume $v_{j+1} \in A_2 \cup A_4$ then v_{j+1} is either in A_2 or in A_4. If $v_{j+1} \in A_2$, $d(v_{j-1}, A_2) = 1$ and if $v_{j+1} \in A_4$, $d(v_{j-1}, A_4) = 1$. Which results in a contradiction. Similarly $v_{j+2} \in A_2 \cup A_4$ leads to contradiction. Hence $v_{j+1}, v_{j+2} \in A_1 \cup A_3$.

If $v_{j+1}, v_{j+2} \in A_1$, then $r(v_{j+1}|\Lambda) = (0, k, 1, k) = r(v_{j+2}|\Lambda)$ results in a contradiction. If v_{j+1} is in A_1 and v_{j+2} in A_3, then $r(v_j|\Lambda) = (1, k, 0, k) = r(v_{j+2}|\Lambda)$ results in a contradiction. Similar arguments work if we either choose 3^{rd} or 4^{th} column of Table 10 for $d(v, A_2)$ or $d(v, A_4)$.

Case 1.2: If $k \leq a \leq k + 1$ and $a' < k$.

For $d(v, A_2)$ we will have Tables 9 to 11 and $d(v, A_4)$ distances are chosen either from Table 12 or from Table 13. It can be verified easily that in all possible choices we will get at least

TABLE 8. $r(v|\Lambda)$ for $n = 8\alpha + 9$

Distances of vertices from:	A_1	A_2	A_3	A_4	A_5
$v_{4l+1}(0 \leq \delta \leq \alpha - 1)$	0	$\delta + 2$	$\delta + 1$	$\delta + 1$	$\delta + 1$
$v_{4\delta+2}(0 \leq \delta \leq \alpha - 1)$	0	$\delta + 2$	$\delta + 1$	$\delta + 1$	$\delta + 2$
$v_{4\delta+3}(0 \leq \delta \leq \alpha - 2)$	0	$\delta + 3$	$\delta + 1$	$\delta + 1$	$\delta + 2$
$v_{4\delta+4}(0 \leq \delta \leq \alpha - 2)$	0	$\delta + 3$	$\delta + 2$	$\delta + 1$	$\delta + 2$
$v_{4\alpha-1}$	0	α	α	α	$\alpha + 1$
$v_{4\alpha}$	0	α	$\alpha + 1$	α	$\alpha + 1$
$v_{8\alpha-4\delta-3}(0 \leq \delta \leq \alpha - 1)$	0	$\delta + 1$	$\delta + 2$	$\delta + 2$	$\delta + 2$
$v_{8\alpha-4\delta-2}(0 \leq \delta \leq \alpha - 1)$	0	$\delta + 1$	$\delta + 1$	$\delta + 2$	$\delta + 2$
$v_{8\alpha-4\delta-1}(1 \leq \delta \leq \alpha - 1)$	0	δ	$\delta + 1$	$\delta + 2$	$\delta + 2$
$v_{8\alpha-1}$	1	0	1	2	2
$v_{8\alpha}$	1	0	1	1	2
$v_{8\alpha+1}$	1	0	1	1	1
$v_{8\alpha+3}$	2	0	1	1	1
$v_{8\alpha+2}$	1	1	0	1	1
$v_{8\alpha+8}$	1	2	0	1	1
$v_{8\alpha+4}$	2	1	1	0	1
$v_{8\alpha+7}$	1	1	1	0	1
$v_{8\alpha+9}$	1	2	1	0	1
$v_{8\alpha+5}$	2	1	1	1	0
$v_{8\alpha+6}$	1	1	1	1	0

two same representations. In Table 12 and 13, we take $\lambda = a$ for $d(v, A_2)$ and $\lambda = a'$ for $d(v, A_4)$. In case of $r = 3$, if we choose 3^{rd} column from Table 10 and 2^{nd} column from Table 12 the representations might not repeat. So following the same procedure as in case (i) we will get $r(v_j|\Lambda) = (1, k, 0, \lambda - 1)$ and $v_{j+1} \notin A_2 \cup A_4$. So either $v_{j+1} \in A_1$ or A_3 so assume that $v_{j+1} \in A_1$, which implies $r(v_{j-1}|\Lambda) = (0, k, 1, \lambda - 1) = r(v_{j+1}|\Lambda)$. If $v_{j+1} \in A_3$ then $r(v_j|\Lambda) = (1, k, 0, \lambda - 1) = r(v_{j+1}|\Lambda)$. So in both cases we get contradiction. A similar argument can be given if we choose distances from Table 13 and Table 10.

Case 1.3: If $a < k$ and $a' < k$.

$d(v, A_2)$ and $d(v, A_4)$ will be chosen from Table 12 or Table 13. It can be verified easily that in all possible cases at least two representations will be same which results in a contradiction.

Case 2: If three vertices of V^* are in the set A_1 i.e. $|V^* \cap A_1| = 3$. We can assume that v_p, v_q, v_r are in $V^* \cap A_1$ and remaining one vertex v_s is in $V^* \cap A_2$. This will give $r(v_p|\Lambda) = (0, 1, 1, a), r(v_q|\Lambda) = (0, 1, 1, b), r(v_r|\Lambda) = (0, 1, 1, c)$.

If $d(v_s, A_4) = \mu$ then either $\mu - 1 \leq a, b, c \leq \mu$ or $\mu \leq a, b, c \leq \mu + 1$.

as v_p, v_q, v_r, v_s are consecutive vertices with connection set $\{1, 2, 3, 4\}$. So by Pigeonhole principle at least two of the vertices will have the same partition representation. Which results in a contradiction.

Case 3: If two vertices of V^* are in the set A_1. i.e. $|V^* \cap A_1| = 2$.

Case 3.1: Assume that v_p, v_q are in $V^* \cap A_1$, v_r in $V^* \cap A_2$ and v_s in $V^* \cap A_4$ then $r(v_p|\Lambda) = (0, 1, 1, 1), r(v_q|\Lambda) = (0, 1, 1, 1)$, results in a contradiction.

Case 3.2: Assume that v_p, v_q are in $V^* \cap A_1$ and v_r, v_s are in $V^* \cap A_2$ then

$r(v_p|\Lambda) = (0, 1, 1, 1) = r(v_q|\Lambda)$ and $r(v_r|\Lambda) = (1, 0, 1, 1) = r(v_s|\Lambda)$.

Which results in a contradiction.

□

TABLE 9. Possible choices for $d(v, A_2)$ and $d(v, A_4)$ where $v \in V^*$ and $r = 2$

v_{j-4}	$k+1$	k	k	k	k
v_{j-3}	k	$k+1$	k	k	k
v_{j-2}	k	k	$k+1$	k	k
v_{j-1}	k	k	k	$k+1$	k

TABLE 10. Possible choices for $d(v, A_2)$ and $d(v, A_4)$ where $v \in V^*$ and $r = 3$

v_{j-4}	$k+1$	$k+1$	k	k	k	k
v_{j-3}	k	$k+1$	$k+1$	k	k	k
v_{j-2}	k	k	$k+1$	$k+1$	k	k
v_{j-1}	k	k	k	$k+1$	$k+1$	k

TABLE 11. Possible choices for $d(v, A_2)$ and $d(v, A_4)$ where $v \in V^*$ and $r \geq 4$

v_{j-4}	$k+1$	$k+1$	$k+1$	$k+1$	k	k	k	k
v_{j-3}	k	$k+1$	$k+1$	$k+1$	$k+1$	k	k	k
v_{j-2}	k	k	$k+1$	$k+1$	$k+1$	$k+1$	k	k
v_{j-1}	k	k	k	$k+1$	$k+1$	$k+1$	$k+1$	k

TABLE 12. Possible choices for $d(v, A_2)$ and $d(v, A_4)$ where $v \in V^*$

v_{j-4}	λ	λ	λ	λ
v_{j-3}	$\lambda-1$	λ	λ	λ
v_{j-2}	$\lambda-1$	$\lambda-1$	λ	λ
v_{j-1}	$\lambda-1$	$\lambda-1$	$\lambda-1$	λ

TABLE 13. Possible choices for $d(v, A_2)$ and $d(v, A_4)$ where $v \in V^*$

v_{j-4}	β	β	β	β
v_{j-3}	$\beta+1$	β	β	β
v_{j-2}	$\beta+1$	$\beta+1$	β	β
v_{j-1}	$\beta+1$	β +1	$\beta+1$	β

The subsequent lemma will be helpful in proving the partition dimension of G_{10}.

LEMMA 3.1. *Let $\Lambda = \{A_1, A_2, A_3, A_4, A_5\}$ be a resolving partition of G_{10}.*

(i) *If $|A_j| = 1$ for some $1 \leq j \leq 5$, then $d(v, A_j) = 2$ for exactly one $v \in V(G_{10})$.*
(ii) *If $|A_j| \geq 2$ for some $1 \leq j \leq 5$, then for all $v \in V(G_{10})$, we have $d(v, A_j) \leq 1$.*

Proof. (i) Let $A_j = \{v_i\}$ for some $1 \leq j \leq 5$, then $d(v_{i+1}, A_j) = d(v_{i+2}, A_j) = d(v_{i+3}, A_j) = d(v_{i+4}, A_j) = d(v_{i-1}, A_j) = d(v_{i-2}, A_j) = d(v_{i-3}, A_j) = d(v_{i-4}, A_j) = 1$ and $d(v_{i+5}, A_j) = 2$.
(ii) If $|A_j| \geq 2$ for some $1 \leq j \leq 5$, then all the vertices in $V(G_{10}) \setminus A_j$ are at distance 1 from some vertex in A_j.

□

THEOREM 3.3. $pd(G_{10}) = 6$.

Proof. Let $A_1 = \{v_0\}, A_2 = \{v_1, v_2, v_3, v_4\}, A_3 = \{v_5, v_8\}, A_4 = \{v_6\}, A_5 = \{v_7\}, A_6 = \{v_9\}$. Since $\Lambda = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ is a resolving partition of $V(G_{10})$, we have $pd(G_{10}) \leq 6$.

By Theorem 3.2 we know that $pd(G_{10}) \geq 5$. We only need to show that $pd(G_{10}) \neq 5$. Let $\Lambda = \{A_1, A_2, A_3, A_4, A_5\}$ be a resolving partition of $V(G_{10})$. Here we have the subsequent cases.

Case 1: If $|A_j| = 2$ for all $j \in \{1, 2, 3, 4, 5\}$. It is clear from Lemma 3.1 that $d(v, A_j) \leq 1$ for all $v \in V(G_{10})$. Therefore, $r(v|\Lambda) = (0, 1, 1, 1, 1)$ for both vertices in A_1. Which contradicts our assumption.

Case 2: If $|A_j| \geq 3$ for some $j \in \{1, 2, 3, 4, 5\}$, consider $|A_1| \geq 3$. Let $x_1, x_2, x_3 \in A_1$. Since the partition representation of x_1, x_2

and x_3 are distinct therefore, there exist $i, j \in \{1, 2, 3\}$ such that $r(v_i|\Lambda))$ and $r(v_j|\Lambda)$ have 2 as one of its coordinates. We can consider, $x_1 \in A_1$ with $d(x_1, A_4) = 2$ and $x_2 \in A_1$ with $d(x_2, A_5) = 2$. Lemma 3.1 implies that all other vertices of G_{10} have the representations with fourth and fifth coordinates at most 1. Since $r = 2$ for G_{10} so there is only one vertex at the diameter distance from any given vertex. This implies that $r(x_1|\Lambda) = (0, 1, 1, 2, 1), r(x_2|\Lambda) = (0, 1, 1, 1, 2)$.

Moreover there is exactly one vertex in G_{10} with the representation having the fifth coordinate 0 and at most two vertices with the representation having fourth coordinate 0. Thus G_{10} contains at least five vertices, say u_1, u_2, u_3, u_4, u_5 with the representations having fourth and fifth coordinates equal to 1. Let $V^* = \{u_1, u_2, u_3, u_4, u_5\}$. We distinguish the subcases.

Case 2.1: Four vertices of V^* are in A_1 or A_2 or A_3.

We can assume that $u_1, u_2, u_3, u_4 \in V^* \cap A_1$ then $r(u_1|\Lambda) = (0, b_1, c_1, 1, 1), r(u_2|\Lambda) = (0, b_2, c_2, 1, 1), r(u_3|\Lambda) = (0, b_3, c_3, 1, 1)$ and $r(u_4|\Lambda) = (0, b_4, c_4, 1, 1)$
where $b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4 \in \{1, 2\}$.

Case 2.1.1: If $b_1 = 2$ or $c_1 = 2$. Suppose $b_1 = 2$ then we must have $c_1 = 1$ as $r = 2$ and Lemma 3.1 implies that $b_2 = b_3 = b_4 = 1$. Also only one of c_2, c_3 and c_4 can be 2. Assume that $c_2 = 2$ then we must have $c_3 = c_4 = 1$. This means that u_3 and u_4 will have same representations, which results in a contradiction.

Case 2.1.2: Suppose $b_1 = 1$ and $c_1 = 1$ then only one of the coordinates of u_2, u_3 and u_4 can be 2. Suppose $b_2 = 2$ then we must have $c_2 = 1$ as $r = 2$ and Lemma 3.1 implies that $b_3 = b_4 = 1$. Also only one of c_3 and c_4 can be 2. Assume that $c_3 = 2$ then $c_4 = 1$. This means u_1 and u_4 will have same representations, which results in a contradiction.

Case 2.2: Three vertices of V^* are in A_1 or A_2 or A_3 and two vertices in one of the other sets. Suppose u_1, u_2, u_3 are in $V^* \cap A_1$ and u_4, u_5 in $V^* \cap A_2$ then $r(u_1|\Lambda) = (0, b_1, c_1, 1, 1)$, $r(u_2|\Lambda) = (0, b_2, c_2, 1, 1)$, $r(u_3|\Lambda) = (0, b_3, c_3, 1, 1)$
$r(u_4|\Lambda) = (a_1, 0, c_4, 1, 1)$ and $r(u_5|\Lambda) = (a_2, 0, c_5, 1, 1)$
Since $|A_1| \geq 3$ and $|A_2| \geq 2$, so by Lemma 3.1 we must have
$a_1 = a_2 = b_1 = b_2 = b_3 = 1$ and only one of c_1, c_2, c_3, c_4 and c_5 can be 2.
So assume that $c_1 = 2$ then $c_2 = c_3 = c_4 = 1$ which means u_2 and u_3 will have same representations, which results in a contradiction. Now if we take $c_4 = 2$ then u_1, u_2 and u_3 will have same representations again we get a contradiction.

Case 2.3: Two vertices of V^* are in A_1 and three in A_2. Suppose u_1, u_2 are in $V^* \cap A_1$ and u_3, u_4, u_5 are in $V^* \cap A_2$ then
$r(u_1|\Lambda) = (0, b_1, c_1, 1, 1), r(u_2|\Lambda) = (0, b_2, c_2, 1, 1)$
$r(u_3|\Lambda) = (a_1, 0, c_3, 1, 1), r(u_4|\Lambda) = (a_2, 0, c_4, 1, 1)$ and
$r(u_5|\Lambda) = (a_3, 0, c_5, 1, 1)$
Since $|A_1| \geq 3$ and $|A_2| \geq 3$, so by Lemma 3.1 we must have
$a_1 = a_2 = a_3 = b_1 = b_2 = 1$ and only one of c_1, c_2, c_3, c_4 and c_5 can be 2.
Assume that $c_1 = 2$ then $c_2 = c_3 = c_4 = c_5 = 1$ which means u_3, u_4 and u_5 will have same representations, which results in a contradiction. Now if we take $c_3 = 2$ then u_1 and u_2 will have same representations and also u_4 and u_5 will have same representations. Again we get a contradiction.

Case 2.4: One vertex of V^* is in A_1, two in A_2 and two in A_3. Suppose u_1 is in $V^* \cap A_1$, u_2, u_3 are in $V^* \cap A_2$ and u_4, u_5 are in $V^* \cap A_3$ then
$r(u_1|\Lambda) = (0, b_1, c_1, 1, 1), r(u_2|\Lambda) = (a_1, 0, c_2, 1, 1)$
$r(u_3|\Lambda) = (a_2, 0, c_3, 1, 1), r(u_4|\Lambda) = (a_3, b_2, 0, 1, 1)$ and
$r(u_5|\Lambda) = (a_4, b_3, 0, 1, 1)$
Since $|A_1| \geq 3, |A_2| \geq 2$ and $|A_3| \geq 2$, so by Lemma 3.1 we must have
$a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = c_1 = c_2 = c_3 = 1$.
Which will give at least two same representations, which results in a contradiction.

Case 2.5: Two vertices of V^* are in each of A_1 and A_2 and one in A_3. Suppose u_1, u_2 are in $V^* \cap A_1$, u_3, u_4 are in $V^* \cap A_2$ and u_5 is in $V^* \cap A_3$ then
$r(u_1|\Lambda) = (0, b_1, c_1, 1, 1)$, $r(u_2|\Lambda) = (0, b_2, c_2, 1, 1)$, $r(u_3|\Lambda) = (a_1, 0, c_3, 1, 1)$,
$r(u_4|\Lambda) = (a_2, 0, c_4, 1, 1)$ and $r(u_5|\Lambda) = (a_3, b_3, 0, 1, 1)$.
Since $|A_1| \geq 3, |A_2| \geq 2$, so by Lemma 3.1 we must have
$a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 1$ and only one of c_1, c_2 and c_3 can be 2 so as in the previous case we will get at least two same representations, which results in a contradiction.

Case 2.6: Three vertices of V^* are in A_2 and two in A_3. Suppose u_1, u_2, u_3 are in $V^* \cap A_2$ and u_4, u_5 are in $V^* \cap A_3$ then
$r(u_1|\Lambda) = (a_1, 0, c_1, 1, 1), r(u_2|\Lambda) = (a_2, 0, c_2, 1, 1)$
$r(u_3|\Lambda) = (a_3, 0, c_3, 1, 1), r(u_4|\Lambda) = (a_2, b_1, 0, 1, 1)$ and
$r(u_5|\Lambda) = (a_3, b_2, 0, 1, 1)$
Since $|A_1| \geq 3, |A_2| \geq 3$ and $|A_3| \geq 2$, so by Lemma 3.1 we must have

$a_1 = a_2 = a_3 = b_1 = b_2 = c_1 = c_2 = c_3 = 1$.
Which will give at least two same representations, which results in a contradiction. So in each case we concluded that $pd(G_{10}) \neq 5$. Hence $pd(G_{10}) = 6$.

□

4. Conclusion

In this article, we concluded that

$$pd(G_n) = \begin{cases} n, & \text{if } 8 \leq n \leq 9; \\ 6, & \text{if } n = 10; \\ 5, & \text{if } n \geq 11. \end{cases}$$

Here we conclude with the following open problem.

OpenProblem 4.1. *Calculate the $pd(C_n(1, 2, \ldots, t))$ for positive integer n and $t \geq 5$.*

Acknowledgement:
The authors are grateful to the reviewer's valuable comments that improved the manuscript.

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