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THE INFLUENCE OF DEBORAH NUMBER ON SOME COUETTE FLOWS OF A MAXWELL FLUID

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Some Couette flows of a Maxwell fluid caused by the bottom plate applying shear rate on the fluid, are studied. Exact expressions for velocity and shear stress corresponding to the fluid motion are determined using Laplace transform. Two particular cases of constant shear rate on the bottom plate and sinusoidal oscillations of the wall shear rate are discussed. Some important characteristics of fluid motion are highlighted through graphs.

Keywords: Couette flow; velocity; shear stress; Maxwell fluid; Deborah number.

1. Introduction

Some fluids are observed to exhibit both characteristics of viscosity and elasticity. Such fluids, known as Maxwell fluids, comprise a bigger subclass of rate type fluids. The constitutive relation corresponding to Maxwell fluids is given by [Bird, 1987],[Bohme, 1987],[Joseph, 1990]

$$\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu\mathbf{A},$$

where $\mathbf{S}$ is the extra stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Erickson tensor, λ and μ are the relaxation time and dynamic viscosity, respectively and the superposed dot indicates the material time derivative. Some viscoelastic fluids showing characteristics of stress relaxation, creep and normal stress differences, being developed in simple shear flows, are better represented by a rather larger class of fluids, known as Oldroyd-B fluids. The constitutive equation corresponding to Oldroyd-B fluids involves relaxation time as well as retardation time, λ_r. Maxwell model can also be considered as a special case of Oldroyd-B model and this case can be achieved by making $\lambda_r \to 0$ in Oldroyd-B fluids constitutive

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equation. Maxwell fluid model has been the subject of study for many researchers because of its simple and convenient approach in regards to determining analytical solutions for various fluid motion problems. In course of time, the motion of Maxwell fluids [Riccius et al., 1987], [Renardy 1986, 1988, 1990, 2000], [Böhme, 2000], [Fetecau and Fetecau, 2005, 2011], [Hayat et al.,] has been studied in various circumstances e.g. fluid owing its motion to the motion of boundary, application of a body force, imposition of pressure gradient and application of tangential shear. The first exact solution of Rayleigh's Stokes' problem for Maxwell fluids was given by [Tanner, 1962]. Some other interesting solutions of Stokes' first problem corresponding to Maxwell fluids have been determined by [Jordan et al., 2004] and [Jordan and Puri, 2005]. Christov [2010] has proposed convincing results corresponding to Stokes' first problem for Oldroyd-B fluid.
The flow of a fluid is called Couette flow if the fluid is bounded by two parallel walls such that they are in relative motion. The flow between two parallel plates such that one plate is at rest and the other one is moving in its plane with a constant speed, is called the simple Couette flow. The flow between two plates produced by a constant pressure gradient in the direction of the flow is termed as Poiseuille flow. The generalized Couette flow is a superposition of the simple Couette flow over Poiseuille flow [Schlichting, 1968]. Some practical applications of this type of flows have been presented in the reference [Erdogan, 1998]. Recently, considerable amount of work regarding Couette flow problem has been done. Siddiqui et al. [2005] considered the problem of steady plane Couette flows between two parallel plates sliding with respect to each other. Asghar et al. [2009] studied the behavior of unsteady Couette flow for second grade fluids. Jha [2001] analyzed natural convection effects on fluid in an unsteady MHD Couette flow. Marques et al. [2000] brought into light the effects of fluid slip at the boundary for Couette flow under steady state conditions. Khalid and Vafai [2004] have studied the effect of slip condition on Couette flows due to an oscillating wall. Denn and Porteous [1971] have presented interesting results regarding unsteady Couette flow of a Maxwell fluid between two infinite parallel plates while similar solution corresponding to second grade fluids were established by [Jordan and Puri, 2002] and [Jordan, 2005]. Some interesting results regarding Couette or Stokes' flows of non-Newtonian fluids can be found in references [Hayat et al., 2007],[Hayat and Javed, 2011],[Danish and Kumar, 2012].
In this paper, we have dealt with Couette flows of a Maxwell fluid caused by the bottom plate which applies on the fluid a shear rate of the form $\frac{\partial u(0,t)}{\partial y} = \frac{\tau_0}{\mu} f(t)$. Similar solutions of the same generality have been recently obtained by [Fetecau et al., 2011] for second grade fluids. Laplace transform has been used to determine exact expressions for velocity and shear stress corresponding to the fluid motion. In particular, the cases of constant shear rate on the bottom plate and sinusoidal oscillations of the wall shear rate are studied. Some relevant properties of velocity and shear stress are brought to light through graphical illustrations.

2. Problem Formulation and Calculation of the Velocity Field

Let us consider an incompressible, homogeneous Maxwell fluid between two flat, infinite solid plates situated in the planes y=0 and y=h of a Cartesian coordinate system Oxyz with the positive y-axis in the upward direction, Fig. 1.

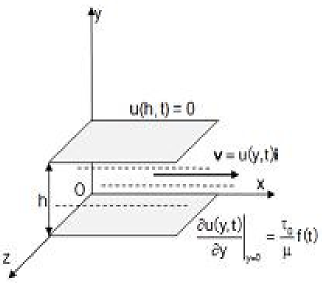

Fig. 1. Geometry flow

Initially, both the fluid and the plates are considered to be at rest. At the moment $t = 0^+$ the motion of the fluid is caused by the bottom plate that applies a shear stress of the form $\frac{\tau_0}{\mu} f(t)$ to the fluid. Here f(t) is a piecewise continuous function defined on [0,∞) and f(0)=0. We also assume that the Laplace transform of function f(t) exists.
For the present fluid motion problem, the velocity vector has the form [Jordan et al., 2004],[Jordan and Puri, 2005]

$$\mathbf{V} = u(y,t)\mathbf{i}. \tag{2.1}$$

while the constitutive and governing equations are

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}, \tag{2.2}$$

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial u(y,t)}{\partial t} = \nu \frac{\partial^2 u(y,t)}{\partial y^2}, \quad (y,t) \in (0,h) \times (0,\infty), \tag{2.3}$$

where $\tau(y,t)$ is the tangential shear stress, μ is the dynamic viscosity of the fluid and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, ρ being the constant density of the fluid.
Also, the initial and boundary conditions are given by

$$u(y,0) = 0, \ \ \frac{\partial u(y,0)}{\partial t} = 0, \ \ \tau(y,0) = 0, \quad y \in [0,h], \tag{2.4}$$

$$\frac{\partial u(0,t)}{\partial y} = \frac{\tau_0}{\mu} f(t), \ u(h,t) = 0. \tag{2.5}$$

By using the following dimensionless variables and functions

$$y^* = \frac{y}{h}, \ t^* = \frac{\nu t}{h^2}, \ \tau^* = \frac{\tau}{\tau_0}, \ u^* = \frac{u}{h\tau_0/\mu}, \ g(t^*) = f\left(\frac{h^2 t^*}{\nu}\right), \tag{2.6}$$

we obtain the next non dimensional initial boundary value problem (dropping the " $*$ " notation)

$$\left(1 + D\frac{\partial}{\partial t}\right)\tau(y,t) = \frac{\partial u(y,t)}{\partial y}, \ \ (y,t) \in (0,1) \times (0,\infty), \tag{2.7}$$

$$\left(1 + D\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2}, \ \ (y,t) \in (0,1) \times (0,\infty), \tag{2.8}$$

$$\frac{\partial u(0,t)}{\partial y} = g(t), \ u(1,t) = 0, \ t \geq 0, \tag{2.9}$$

$$u(y,0) = 0, \ \ \frac{\partial u(y,0)}{\partial t} = 0, \ \ \tau(y,0) = 0, \ \ y \in [0,1], \tag{2.10}$$

where $D = \frac{\lambda}{\left(h^2/\nu\right)}$ is the Deborah number.
By applying the temporal Laplace transform L [Debnath and Bhatta, 2007], to Eqs. (2.7)-(2.9) and employing the initial conditions (10), we obtain the problem

$$(1 + Dq)\bar{\tau}(y,q) = \frac{\partial \bar{u}(y,q)}{\partial y}, \ \ y \in (0,1), \ Req > 0, \tag{2.11}$$

$$\frac{\partial^2 \bar{u}(y,q)}{\partial y^2} = (Dq^2 + q)\bar{u}(y,q), \ \ y \in (0,1), \ Req > 0, \tag{2.12}$$

$$\frac{\partial \bar{u}(0,q)}{\partial y} = G(q), \ \ \bar{u}(1,q) = 0, \tag{2.13}$$

where $\bar{\tau}(y,q) = L\{\tau(y,t)\}$, $\bar{u}(y,q) = L\{u(y,t)\}$, $G(q) = L\{g(t)\}$ are the Laplace transforms of the functions $\tau(y,t)$, $u(y,t)$ and $g(t)$, respectively.
The transform domain solution of Eq. (2.12) with the boundary conditions (2.13) is given by

$$\bar{u}(y,q) = G(q)G_1(y,q), \tag{2.14}$$

where

$$G_1(y,q) = \frac{sh[(y-1)\sqrt{Dq^2+q}]}{\sqrt{Dq^2+q}ch(\sqrt{Dq^2+q})}. \tag{2.15}$$

In order to find the inverse Laplace transform of the right part of Eq. (2.14), we consider the auxiliary function

$$F_1(y,q) = \frac{sh[(y-1)\sqrt{q}]}{\sqrt{q}ch(\sqrt{q})}, \tag{2.16}$$

which is the image of the function

$$f_1(y,t) = -2\sum_{n=0}^{\infty} \cos(\alpha_n y)\exp(-{\alpha_n}^2 t), \tag{2.17}$$

with $\alpha_n = \frac{(2n+1)\pi}{2}$, n=0,1,2,...
Since $G_1(y,q) = (F_1 o w)(q) = F_1(y, w(q))$, with $w(q) = Dq^2 + q = D\left(q + \frac{1}{2D}\right)^2 - \frac{1}{4D}$, then its inverse Laplace transform is

$$g_1(y,t) = L^{-1}\{G_1(y,q)\} = \int_0^\infty f_1(y,z)p(z,t)dz, \tag{2.18}$$

where

$$p(z,t) = L^{-1}\{e^{-zw(q)}\} = L^{-1}\left\{e^{\frac{z}{4D}}.e^{-zD\left(q+\frac{1}{2D}\right)^2}\right\}. \tag{2.19}$$

By using Eq. $(A.1)$ from Appendix, we obtain

$$p(z,t) = \frac{t}{2}e^{\frac{z-2t}{4D}}\sum_{k=0}^{\infty}\frac{(-Dz)^k}{(k+1)!(2k+1)!}\int_0^\infty J_2(2\sqrt{xt})dx, \tag{2.20}$$

where $J_\nu(.)$ is the Bessel function of first kind and order ν.

Replacing (2.17) and (2.20) into (2.18) we find that

$$g_1(y,t) = -te^{-\frac{t}{2D}}\sum_{n=0}^{\infty}\cos(\alpha_n y)\sum_{k=0}^{\infty}\frac{(-Dz)^k}{(k+1)!(2k+1)!}\int_0^\infty x^{2k+1}J_2(2\sqrt{xt})dx\int_0^\infty z^k e^{-\left({\alpha_n}^2-\frac{1}{4D}\right)z}dz$$

$$= -te^{-\frac{t}{2D}}\sum_{n=0}^{\infty}\cos(\alpha_n y)\int_0^\infty J_2(2\sqrt{xt})\sum_{k=0}^{\infty}\frac{(-D)^k\Gamma(k+1)x^{2k+1}}{(k+1)!(2k+1)!b_n^{k+1}}dx, \tag{2.21}$$

where $b_n = {\alpha_n}^2 - \frac{1}{4D} > 0$ and Γ is the Gamma function.
By using Eq. $(A.2)$ from Appendix, we obtain the following simpler expression of the function $g_1(y,t)$:

$$g_1(y,t) = -\frac{2t}{D}e^{-\frac{t}{2D}}\sum_{n=0}^{\infty}\cos(\alpha_n y)\int_0^\infty J_2(2\sqrt{xt})\frac{1}{x}\left[1-\cos\left(x\sqrt{\frac{D}{b_n}}\right)\right]dx. \tag{2.22}$$

Now, using the properties of the Bessel functions [Abramowitz and Stegun, 1972], we obtain

$$g_1(y,t) = -2e^{-\frac{t}{2D}} \sum_{n=0}^{\infty} \frac{\cos(\alpha_n y)}{\sqrt{b_n D}} \sin\left(t\sqrt{\frac{b_n}{D}}\right). \tag{2.23}$$

Finally, using Eqs. (2.14), (2.23) and the convolution theorem we obtain the expression of the velocity given by

$$u(y,t) = (g * g_1)(t) = -2\sum_{n=0}^{\infty} \frac{\cos(\alpha_n y)}{\sqrt{b_n D}} \int_0^t g(t-s)e^{-\frac{s}{2D}} \sin\left(s\sqrt{\frac{b_n}{D}}\right) ds. \tag{2.24}$$

3. Calculation of the Shear Stress

In order to determine the shear stress $\tau(y,t)$, we use Eqs. (2.11), (2.14) and (2.23). Introducing Eq. (2.14) into Eq. (2.11) we find that

$$\bar{\tau}(y,q) = \frac{1}{1+Dq} G(q) \frac{\partial G_1(y,q)}{\partial y} = \frac{G(q)}{D} \frac{1}{q+\frac{1}{D}} \frac{\partial G_1(y,q)}{\partial y} = \frac{1}{D} G(q) G_2(y,q), \tag{3.1}$$

where the function

$$G_2(y,q) = \frac{1}{q+1/D} \frac{\partial G_1(y,q)}{\partial y} \tag{3.2}$$

has the inverse Laplace transform

$$g_2(y,t) = \int_0^t e^{-\frac{(t-s)}{D}} \frac{\partial g_1(y,s)}{\partial y} ds = 2e^{-\frac{t}{D}} \sum_{n=0}^{\infty} \frac{\alpha_n \sin(\alpha_n y)}{\sqrt{b_n D}} \int_0^t e^{\frac{s}{2D}} \sin\left(s\sqrt{\frac{b_n}{D}}\right) ds.$$

By evaluating the last integral, it results

$$g_2(y,t) = e^{-\frac{t}{D}} + 2e^{-\frac{t}{2D}} \sum_{n=0}^{\infty} \frac{\sin(\alpha_n y)}{\alpha_n} \left[\frac{1}{2\sqrt{b_n D}} \sin\left(t\sqrt{\frac{b_n}{D}}\right) - \cos\left(t\sqrt{\frac{b_n}{D}}\right) \right] \tag{3.3}$$

Consequently, the shear stress can be written in the simpler form

$$\tau(y,t) = \frac{1}{D}(g * g_2)(t) = \frac{1}{D} \int_0^t g(t-s) g_2(y,s) ds, \tag{3.4}$$

where $g_2(y,s)$ is given by the above relation.

4. Some particular cases of the motion

In this section we consider the following two expressions of the function $g(t)$

- $g(t) = H(t)$, $H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$ being the Heaviside function;
- $g(t) = sin(\Omega t)$, $\Omega > 0$ is the constant frequency of the oscillations.

In the first case, replacing $g(t-s)=1$ into Eq. (2.24) we obtain

$$u(y,t)=-2\sum_{n=0}^{\infty}\frac{\cos(\alpha_n y)}{\sqrt{b_n D}}\int_0^t e^{-\frac{s}{2D}}\sin\left(s\sqrt{\frac{b_n}{D}}\right)ds$$

$$=-2\sum_{n=0}^{\infty}\frac{\cos(\alpha_n y)}{{\alpha_n}^2}+2e^{-\frac{t}{2D}}\sum_{n=0}^{\infty}\frac{\cos(\alpha_n y)}{{\alpha_n}^2}\left[\frac{1}{2\sqrt{b_n D}}\sin\left(t\sqrt{\frac{b_n}{D}}\right)+\cos\left(t\sqrt{\frac{b_n}{D}}\right)\right].$$

By using Eq. (A.3) from Appendix we get the velocity field under the form

$$u(y,t)=y-1+2e^{-\frac{t}{2D}}\sum_{n=0}^{\infty}\frac{\cos(\alpha_n y)}{{\alpha_n}^2}\left[\frac{1}{2\sqrt{b_n D}}\sin\left(t\sqrt{\frac{b_n}{D}}\right)+\cos\left(t\sqrt{\frac{b_n}{D}}\right)\right] \tag{4.1}$$

The velocity given by Eq. (4.1) has the following temporal limits:

$$\lim_{t\to 0^+}u(y,t)=0, \quad \lim_{t\to\infty}u(y,t)=y-1. \tag{4.2}$$

From Eq. (4.2) it results that the velocity $u(y,t)$ does not exhibit a jump of discontinuity at t=0 and, for $t\to\infty$ it reduces to the "permanent solution" (or steady solution) $u_s=y-1$. Furthermore, all initial and boundary conditions are clearly satisfied.

By using the illustrations generated with the software Mathcad, we discuss some physical aspects of the flow. In all figures we use $\nu=0.1655\,m^2/s$ $\lambda=0.062951s$, $\rho=840\,kg/m^3$. In Fig. 2, we have plotted the profiles of the velocity $u(y,t)$ given by Eq. (4.1), versus $y\in[0,1]$, $t\in\{1,1.5,3\}$ and for different values of the Deborah number D. It is clear that the absolute values of the velocity decrease if the Deborah number decreases.

For large values of the time t the diagrams of the velocity tend to the diagram of the "permanent velocity" $u_p=y-1$. Figure 3 contains diagrams of velocity $u(y,t)$, versus t, for $y\in\{0.1,0.4,0.6\}$ and different values of Deborah number D. For small values of the time t the influence of the Deborah number on the velocity is insignificant. In the interval $t\in[1,4]$ the influence of the Deborah number on the velocity is significant and the velocity increases if the Deborah number decreases. For t$\geq$ 4 the velocity tends to the permanent velocity.

In order to determine the velocity field corresponding to the oscillating shear rate $g(t)=\sin(\Omega t)$ we use Eq. (2.24) with $g(t-s)=\sin\Omega(t-s)$ and have

$$u(y,t)=u_p(y,t)+u_t(y,t) \tag{4.3}$$

where

$$u_p(y,t)=2\Omega\cos(\Omega t)\sum_{n=0}^{\infty}\frac{\cos(\alpha_n y)}{({\alpha_n}^2-D\Omega^2)^2+\Omega^2}$$

$$-2\sin(\Omega t)\sum_{n=0}^{\infty}\frac{({\alpha_n}^2-D\Omega^2)\cos(\alpha_n y)}{({\alpha_n}^2-D\Omega^2)^2+\Omega^2}, \tag{4.4}$$

$$u_t(y,t) = -\Omega e^{-\frac{t}{2D}} \sum_{n=0}^{\infty} \frac{\cos(\alpha_n y)}{\sqrt{b_n D}[({\alpha_n}^2 - D\Omega^2)^2 + \Omega^2]}$$
$$\times \left[2\sqrt{b_n D} \cos\left(t\sqrt{\frac{b_n}{D}} \right) + [1 - 2D({\alpha_n}^2 - D\Omega^2)] \sin\left(t\sqrt{\frac{b_n}{D}} \right) \right]. \tag{4.5}$$

The permanent solution (4.4) can also be written in the simpler form

$$u_p(y,t) = \frac{1}{(A^2+B^2)(sh^2 A + cosh^2 B)} \begin{Bmatrix} [AM_2(y) - BM_1(y)]\cos(\Omega t) \\ +[AM_1(y) + BM_2(y)]\sin(\Omega t) \end{Bmatrix}, \tag{4.6}$$

where

$$2A^2 = \Omega\sqrt{D^2\Omega^2+1} - D\Omega^2; \quad 2B^2 = \Omega\sqrt{D^2\Omega^2+1} + D\Omega^2, \tag{4.7}$$

$$M_1(y) = chA \cos B sh[A(y-1)] \cos[B(y-1)] + shA \sin B ch[A(y-1)] \sin[B(y-1)], \tag{4.8}$$

$$M_2(y) = chA \cos B ch[A(y-1)] \sin[B(y-1)] - shA \sin B sh[A(y-1)] \cos[B(y-1)]. \tag{4.9}$$

Physical aspects of the flow in the case of sinusoidal shear rate on the bottom plate are illustrated by means of the figures 4 and 5.

In Fig. 4 we plotted the velocity $u(y,t)$ given by Eq. (4.3), versus y, for $\Omega = 2$, $t \in \{5, 10, 15\}$ and for different values of the Deborah number D. As shown in these diagrams, for a fixed time t, the influence of the Deborah number on the velocity can be different. For example, for $t \in \{5, 15\}$ the velocity increases if the Deborah number decreases, and for $t = 10$ the velocity decreases if the Deborah number decreases.

Fig. 5 contains the profiles of the starting velocity u(y,t) given by Eq. (4.3) and the "permanent solution" given by Eq. (4.4).These diagrams were plotted versus t, for $y = 0.5$, $\Omega \in \{0.5, 1.2\}$ and for different values of the Deborah number D. An important practical aspect of this type of flow is the achievement of "steady-state" flow. In this case the flow is in accordance with the permanent solution and it achieves after a time t from which the transient solution can be neglected. It is clear that, for given values of the frequency of oscillations of the shear rate, the time to reach the steady-state is decreasing if the Deborah number decreases. Also, this time decreases if the frequency Ω increases.

5. Conclusions

The aim of this paper is to find some new exact solutions for Couette flows of a Maxwell fluid generated by a time-dependent shear rate given on the bottom plate. Expressions for velocity and shear stress are obtained for the general case

$\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = \frac{\tau_0}{\mu} f(t)$. Two particular cases corresponding to a constant shear rate and sinusoidal oscillations of the shear rate are analyzed. The influence of the Deborah number on the fluid motion was studied by means of numerical and graphical results generated with the software Mathcad. The time to reach the steady-state flow can also be obtained by graphical illustrations. The dependence of this time of the Deborah number was also studied.

Appendix

The following relations are used:

$$L^{-1}\left\{\left(q+\frac{1}{2D}\right)^2 e^{-zD\left(q+\frac{1}{2D}\right)^2}\right\} = \frac{e^{\frac{-t}{2D}}}{2}\int_0^\infty J_0(2\sqrt{xt})\sum_{k=0}^\infty \frac{(-Dz)^k x^{2(k+1)}}{(k+1)!(2k+1)!}dx, \qquad (A.1)$$

Also,

$$\int_0^\infty (t-s)J_0(2\sqrt{xs})ds = \frac{t}{x}J_2(2\sqrt{xt}).$$

$$\sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{(k+1)(2k+1)!} = \frac{2}{x}(1-\cos x), \qquad (A.2)$$

$$-\sum_{n=0}^\infty \frac{\cos(\alpha_n y)}{{\alpha_n}^2} = y-1. \qquad (A.3)$$

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Journal reference:

Abramowitz, M. and Stegun, I. A. [1972] (Eds.) 'Modified Bessel Functions I and K, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, 9th printing" (New York: Dover), 374-377.

Asghar, S. , Hayat, T. and Ariel, P. D. [2009] "Unsteady Couette flows in a second grade fluid with variable material properties," *Comm. Non- Lin. Sci. Num.*

Simul. **14**(1), 154-159.

Bird, R. B. [1987]*Dynamics of Polymeric liquids* (vol.1, Fluid Mechanics, Wiley, New York).

Bohme, G. [1987]*Non-Newtonian Fluid Mechanics* (North-Holland, New York).

Christov, I. C. [2010] "Stokes' first problem for some non-Newtonian fluids: Results and mistakes," *Mech. Res. Commun.* **37**, 717-723.

Danish, M. and Kumar, D. S. [2012] "Exact analytical solutions for the Poiseuille and Couette Poiseuille flow of third grade fluid between parallel plates," *Comm. Non-Lin. Sci. Num. Simul.* **17**(3), 1089-1097.

Debnath, L. and Bhatta, D. [2007] *Integral Transforms and their Applications (second edition)* (Chapman and Hall, CRC, London).

Denn, M. M. and Porteous, K. C. [1971] "Elastic effects in flow of visco-elastic liquids," *Chem. Eng. J.* **2**, 280-286.

Erdogan, M. E. [1998] "Effects of the side walls in Generalized Couette flow," *J. Appl. Mech. Eng.* **3**, 271-286.

Fetecau, C. and Fetecau, C. [2005] "On some axial Couette flows of non-Newtonian fluids," *Z. Angew. Math. Phys.* **56**, 1098-1106.

Fetecau, C. and Fetecau, C. and Rana, M. [2011] "General solutions for the unsteady flow of second-grade fluids over an infinite plate that applies arbitrary shear to the fluid," *Z. Naturforsch.* **66a**, 753-759.

Hayat, T. , Khan, M. and Ayub, M. [2007] "," , .

Hayat, T. , Sajid, M. and Ayub, M. [2007] "A note on series solution for generalized Couette flow," *Comm. in Non-Lin. Sci. and Num. Simul.* **12**, 1481-1487.

Hayat, T. and Javed, M. [2011] "Wall properties and slip effects on the magnetohydrodynamic peristaltic motion of a viscous fluid with heat transfer and porous space," *Asia-Pacific J. of Chem. Eng.* **6**(4), 649-658.

Jha, B. k. [2001] "Natural convection in unsteady MHD Couette flow," *Heat and Mass transfer* **37**, 329-331.

Jordan, P. M. and Puri, P. [2002] "Exact solutions for the unsteady plane Couette flow of a dipolar fluid," *Proc. Roy. Soc. London Ser. A 458* **2021**, 1245-1272.

Jordan, P. M. [2005] "A note on start-up, plane Couette flow involving second grade fluids," *Math. Prob. Eng.* **5**, 539-545.

Jordan, P. M. , Puri, A. and Boros, G. [2004] "On a new exact solution to Stokes' first problem for Maxwell fluids," *Int. J. Non-linear Mech.* **39**, 1371-1377.

Jordan, P. M. and Puri, A. [2005] "Revisiting Stokes' first problem for Maxwell fluids," *Q. J. Mech. Appl. Math.* **58**(2), 213-227.
Joseph, D. D. [1990]*Fluid Mechanics of Viscoelastic liquids* (Springer, New York).

Khaled, A. R. A. and Vafai, K. [2004] "The effect of the slip condition on Stokes' and Couette flows due to an Oscillating wall: Exact solutions" *Int. J. Non-Linear Mech.* **39**, 795-809.

Marques, Jr. W. , Kremer, G. M. and Sharipov, F. M. [2000] "Couette flow with slip and jump boundary conditions," *Cont. Mech. Therm.* **12**, 379-386.

Renardy, M. and Renardy, Y. [1986] "Linear stability of plane Couette flow of an upper convected Maxwell fluid," *Jour. of Non-Newtonian Fluid Mech.* **22**, 23-33.

Renardy, M. [1988] "In flow boundary conditions for steady flow of viscoelastic fluids with differential constitutive laws," *Rocky Mount. J. Math.* **18**, 445-453.

Renardy, M. [1990] "An alternative approach to inflow boundary conditions for Maxwell fluids in three space dimensions," *Jour. of Non-Newtonian Fluid Mech.* **36**, 419-425.

Renardy, M. [2000] "Mathematical Analysis of viscoelastic flows," *SIAM* ISBNo 89871-457-5. .

Riccius, O. , Joseph, D. D. and Arney, M. [1987] "Shear-Wave speeds and elastic moduli for different liquids Part 3,n Experiments -update," Rheol. Acta **26**, 96-99.

Schlichting, H. [1968] *Boundary Layer Theory* (McGraw-Hill, New York).

Siddiqui, A. M. , Ahmed, M. Islam, S. and Ghori, Q. K. [2005] "Homotopy analysis of Couette and Poiseuille flows for fourth grade fluids," *Acta Mech.* **180**, 117-132.

Tanner, R. I. [1962] "Note on the Rayleigh problem for a viscoelastic fluid," Z. Angew. Math. Phys. **13**, 573-580.

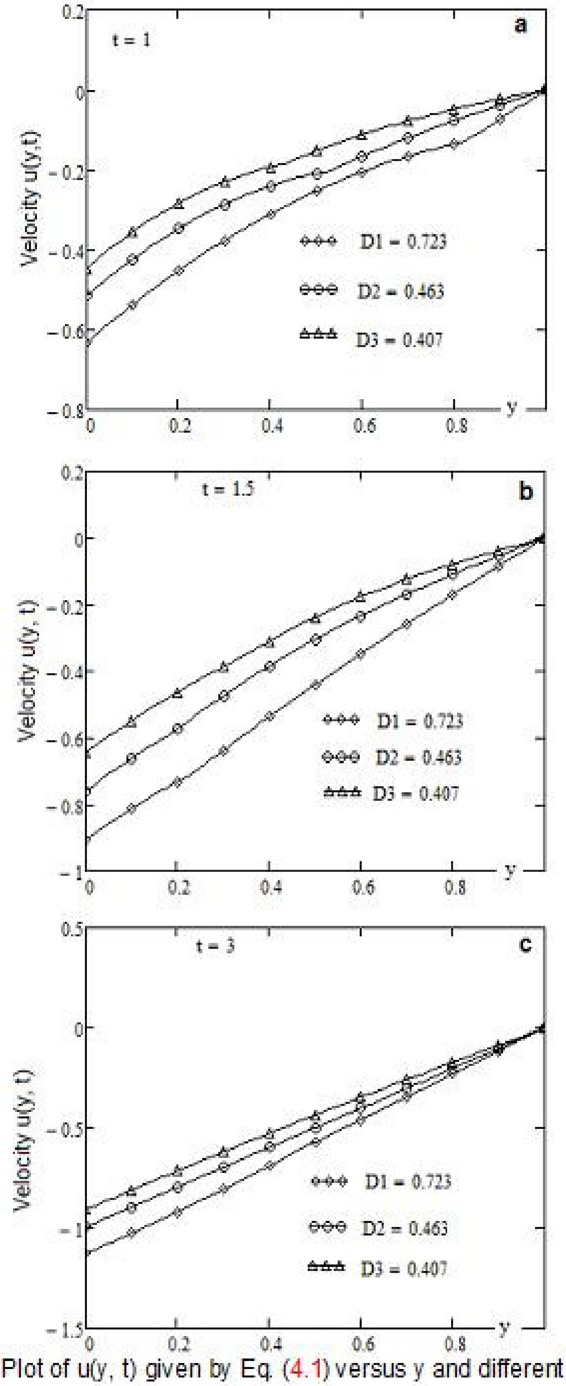

Fig.2 Plot of u(y, t) given by Eq. (4.1) versus y and different values of the Deborah number

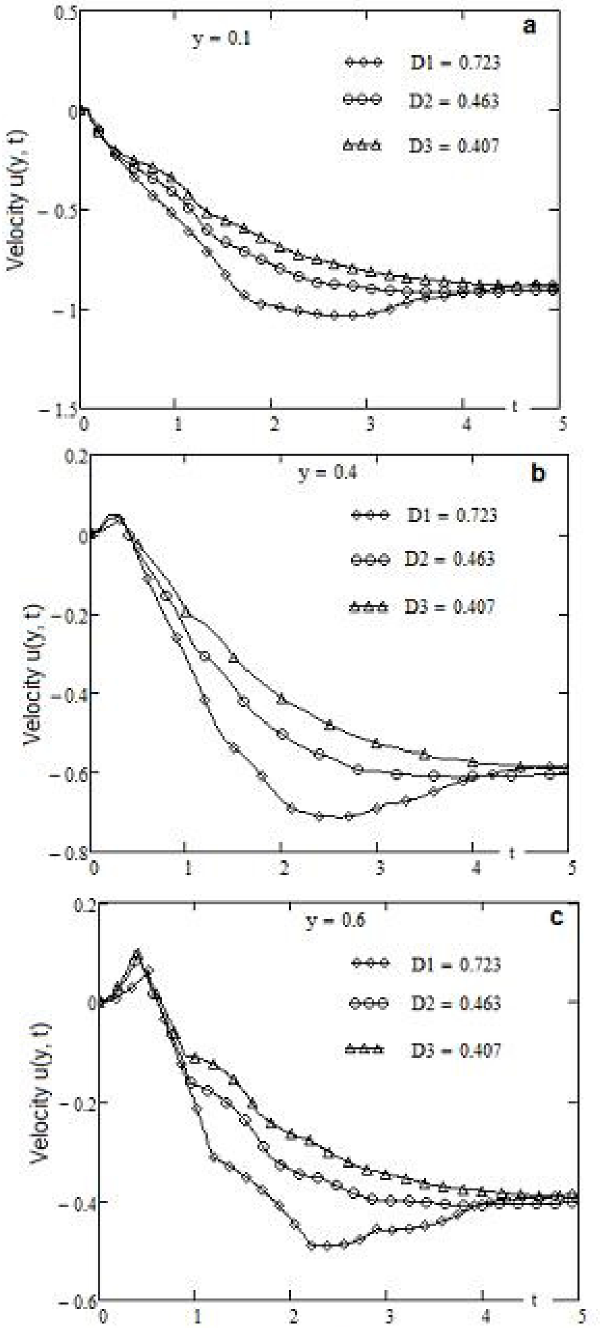

Fig. 3. Plot of u(y, t) given by Eq. (4.1) versus t and different values of the Deborah number

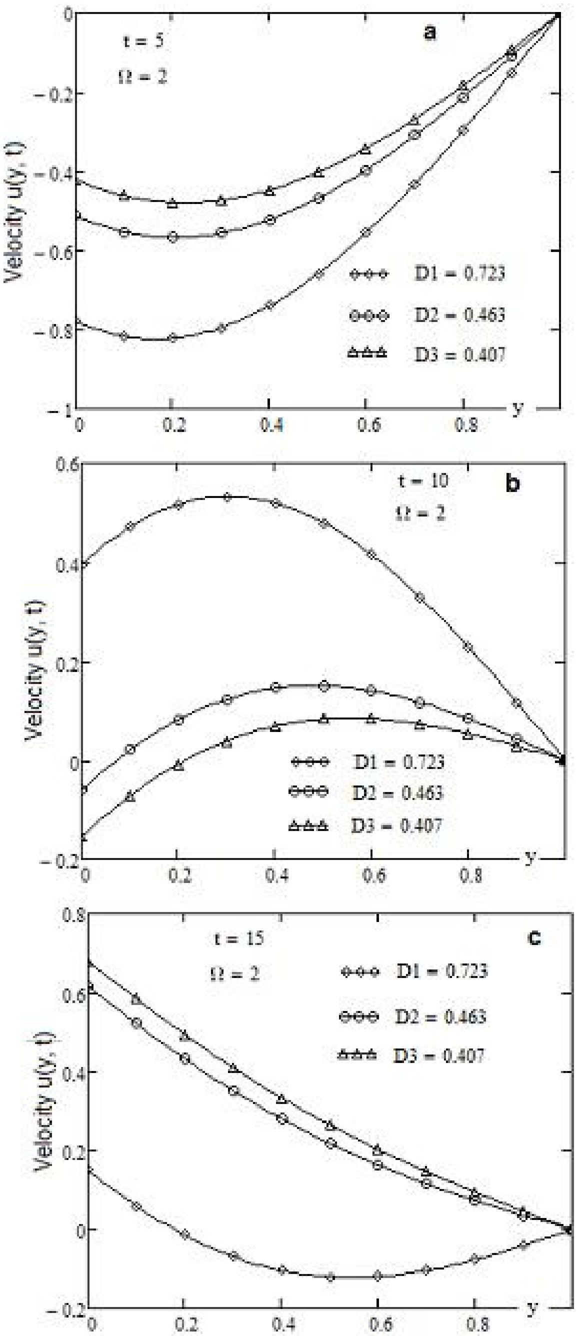

Fig.4 Plot of u(y, t) given by Eq. (4.3) versus y and different values of the Deborah number

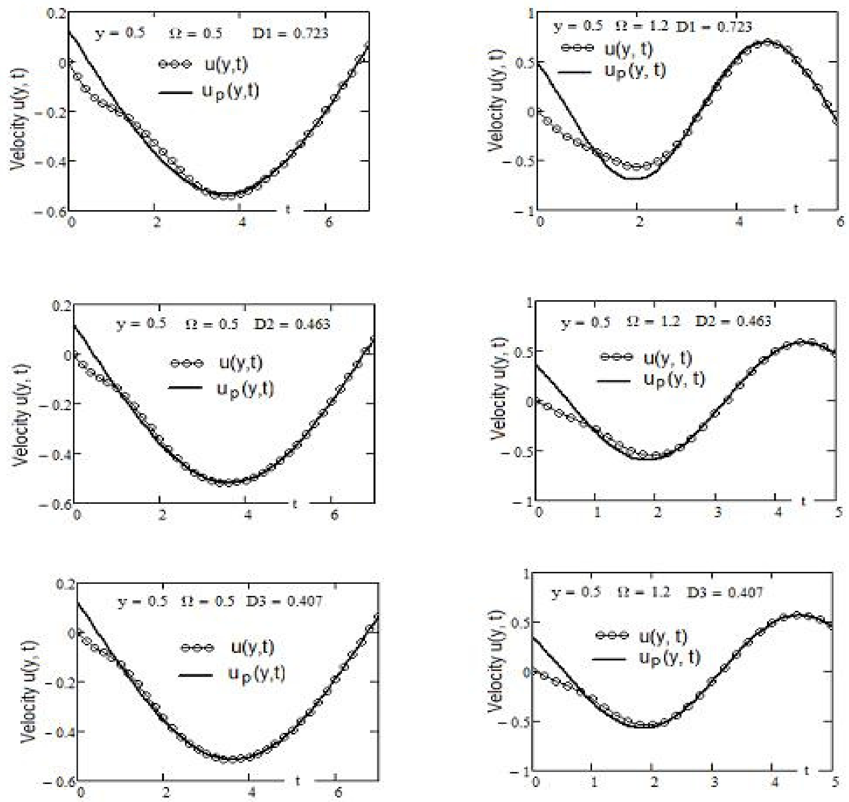

Fig. 5 Plot of the velocity u(y, t) given by Eq. (4.3) and permanent solution $u_p(y, t)$ given by Eq. (4.4), versus t and different values of the Deborah number