

## FRACTIONAL MAGNETOHYDRODYNAMICS OLDROYD-B FLUID OVER AN OSCILLATING PLATE

by

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Original scientific paper  
DOI: 10.2298/110731140J

*This paper presents some new exact solutions corresponding to the oscillating flows of a magnetohydrodynamic Oldroyd-B fluid with fractional derivatives. The fractional calculus approach in the governing equations is used. The exact solutions for the oscillating motions of a fractional magnetohydrodynamic Oldroyd-B fluid due to sine and cosine oscillations of an infinite plate are established with the help of discrete Laplace transform. The expressions for velocity field and the associated shear stress that have been obtained, presented in series form in terms of Fox H-functions, satisfy all imposed initial and boundary conditions. Similar solutions for ordinary magnetohydrodynamic Oldroyd-B, fractional and ordinary magnetohydrodynamic Maxwell, fractional and ordinary magnetohydrodynamic second grade and magnetohydrodynamic Newtonian fluid as well as those for hydrodynamic fluids are obtained as special cases of general solutions. Finally, the obtained solutions are graphically analyzed through various parameters of interest.*

*Mathematics Subject Classification (2010).* 76A05, 76A10

*Key words:* magnetohydrodynamic Oldroyd-B fluid, fractional derivatives, oscillating flows, exact solutions, Fox H-functions, velocity field, shear stress, discrete laplace transforms.

### Introduction

Flows in the neighborhood of spinning or oscillating bodies are of interest to both academic field and industry. The study of time-dependent flows of viscoelastic fluids caused by the oscillations of a flat plate is of considerable interest both industrially as well as a test to assess the performance of numerical methods for the computation of transient flows. Such flows are not only of fundamental and theoretical interest but they also occur in many biological and industrial processes such as the quasi-periodic blood flow in the cardiovascular system, acoustic streaming around an oscillating body, an unsteady boundary layer with fluctuations and so forth. Although exact solutions to various oscillating flow problems of Newtonian fluid have been obtained and are available in the literature, the first closed-form transient solution for the flow of a Newtonian fluid due to an oscillating plate seems to be that of Penton [1]. Puri and Kythe [2] have discussed an unsteady flow problem which deals with non-classical heat condition effects and the structure of waves in Stokes' second problem. Erdogan [3] provided two starting solutions for the motion of a viscous fluid due to cosine and sine oscillations of a flat plate. The first exact solutions for the longitudinal and torsional oscillations of a rod in a non-Newtonian fluid

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are those obtained by Rajagopal [4] and Rajagopal and Bhatnagar [5]. Moreover, exact solutions for various flow geometries in non-Newtonian fluids have been also obtained by Hayat *et al.* [6], Aksel *et al.* [7], Khan *et al.* [8], and Fetecau *et al.* [9, 10]. More recently exact solutions for oscillating flow of different non-Newtonian fluids have been obtained by Khan *et al.* [11, 12], Mahmood *et al.* [13], Zhen *et al.* [14], and Anjum *et al.* [15].

These exact solutions become even rare if the constitutive equations of non-Newtonian fluids with fractional derivatives are considered. For numerous fluids between elastic and viscous materials the fractional constitutive relationship model has an advantage over the customary constitutive relationship model. Fractional calculus has achieved much success in the description of the complex dynamics. The constitutive equations with fractional derivatives have been proved to be a valuable tool to handle viscoelastic properties [16]. In [17], the rest state stability of an objective fractional derivative viscoelastic fluid model has been established which is an important finding that gives strength to the physical basis of these fractional models. Moreover, a very good fit of experimental data is achieved when the constitutive equation with fractional derivatives is used [18]. We mention here some recent contributions [19-35] in the discussion of flows of viscoelastic fluids with a fractional calculus approach.

The aim of this communication is two fold. Firstly, it is to give new exact solutions for magnetohydrodynamic (MHD) Oldroyd-B fluids with fractional derivatives, which is more natural and appropriate tool to describe the complex behavior of non-Newtonian fluids. Secondly, it is to study the oscillating effects on MHD Oldroyd-B fluids with fractional derivatives, which is important due to their practical applications. More precisely, our aim is to find the velocity field and the shear stress corresponding to the motion of fractional MHD Oldroyd-B fluid due to a plate that oscillates with sine or cosine oscillations in its own plane. The general solutions are obtained using the discrete Laplace transform formula for sequential fractional derivatives. They are presented in series form in terms of the Fox H-functions. The similar solutions for ordinary MHD Oldroyd-B, fractional and ordinary MHD Maxwell, fractional and ordinary MHD second grade and MHD Newtonian fluids, can easily be obtained as special cases of general solutions. Furthermore, the solutions for hydrodynamic fluids are also easily obtained from general solutions as special cases and they are similar with previously known results in literature. The general solutions also recover many existing solutions for Stokes' first and second problem for MHD and ordinary non-Newtonian fluids. Finally, the influence of the material, magnetic and fractional parameters on the motion of MHD fractional Oldroyd-B fluids is underlined by graphical illustrations. The difference among fractional MHD Oldroyd-B, fractional MHD Maxwell, fractional MHD second grade and MHD Newtonian fluid models is also spotlighted.

### Basic governing equations

The continuity equation and the equation of motion for the flow of an incompressible fluid, in the absence of body forces, are:

$$\nabla \mathbf{V} = 0, \quad \nabla \mathbf{T} = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\nabla)\mathbf{V} \quad (1)$$

where  $\rho$  is the fluid density,  $\mathbf{V}$  – the velocity field,  $t$  – the time and  $\nabla$  represents the gradient operator. The Cauchy stress  $\mathbf{T}$  in an incompressible Oldroyd-B fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} = \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T)] \quad (2)$$

where  $-p\mathbf{I}$  denotes the indeterminate spherical stress due to the constraint of incompressibility,  $\mathbf{S}$  is the extra-stress tensor,  $\mathbf{L}$  – the velocity gradient,  $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$  – the first Rivlin Ericksen tensor,  $\mu$  – the dynamic viscosity of the fluid,  $\lambda$  and  $\lambda_r$  are relaxation and retardation times, the superposed dot

indicates the material time derivative and the superscript  $T$  indicates the transpose operation. The model characterized by the constitutive eq. (2) contains as special cases the upper-convected Maxwell model for  $\lambda_r \rightarrow 0$  and the Newtonian fluid model for  $\lambda_r \rightarrow 0$  and  $\lambda \rightarrow 0$ . In some special flows, as those to be considered here, the governing equations for an Oldroyd-B fluid resemble for a fluid of second grade. For the problem under consideration we assume a velocity field  $\mathbf{V}$  and an extra-stress tensor  $\mathbf{S}$  of the form:

$$\mathbf{V} = \mathbf{V}(y,t) = u(y,t)i, \quad \mathbf{S} = \mathbf{S}(y,t), \quad (3)$$

where  $i$  is the unit vector along the x-co-ordinate direction. For these flows the constraint of incompressibility is automatically satisfied. In the absence of a pressure gradient in the x-direction, the governing equations of the fractional MHD Oldroyd-B fluid are [26]:

$$(1 + \lambda^\alpha D_t^\alpha) \frac{\partial u(y,t)}{\partial t} = \nu(1 + \lambda_r^\beta D_t^\beta) \frac{\partial^2 u(y,t)}{\partial y^2} - M(1 + \lambda^\alpha D_t^\alpha)u(y,t) \quad (4)$$

$$(1 + \lambda^\alpha D_t^\alpha) \tau(y,t) = \mu(1 + \lambda_r^\beta D_t^\beta) \frac{\partial u(y,t)}{\partial t} \quad (5)$$

where  $\tau(y,t) = S_{xy}(y,t)$  is the non-trivial shear stress,  $\rho$  – the constant density of the fluid,  $\nu = \mu/\rho$  – the kinematic viscosity,  $M = \sigma B_0^2/\rho$ ,  $\sigma$  – the electrical conductivity of the fluid, and  $B_0$  – the applied magnetic field [36-38]. Also  $\alpha$  and  $\beta$  are fractional parameters such that  $0 \leq \beta < 1$  and the fractional differential operator so called Caputo fractional operator  $D_t^p$  defined by [39, 40]:

$$D_t^p f(t) = \begin{cases} \frac{1}{\Gamma(1-p)} \int_0^t \frac{f'(\tau)}{(t-\tau)^p} d\tau, & 0 < p < 1 \\ \frac{df(t)}{dt}, & p = 1 \end{cases} \quad (6)$$

where  $\Gamma(\cdot)$  is the Gamma function. In the following the system of fractional partial differential eqs. (4) and (5), with appropriate initial and boundary conditions, will be solved by means of discrete Laplace transform [25-30].

### Statement of the problem

Consider an incompressible MHD Oldroyd-B fluid with fractional derivatives occupying the space lying over an infinitely extended plate which is situated in the  $(x, z)$  plane and perpendicular to the  $y$ -axis. Initially the fluid, as well as the plate is at rest and at  $t = 0^+$  the plate starts to oscillate in its own plane. Due to the shear, the fluid above the plate is gradually moved. Its velocity is of the form (3)<sub>1</sub> while the governing equations are given by eqs. (4) and (5). The appropriate initial and boundary conditions are:

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0, \quad \tau(y,0) = 0, \quad y > 0 \quad (7)$$

$$u(y,0) = U \sin(\omega t) \quad \text{or} \quad UH(t) \cos(\omega t); \quad t \geq 0 \quad (8)$$

where  $\omega$  is the frequency,  $U$  – the amplitude of the velocity of the plate and  $H(\cdot)$  is the Heaviside function. Furthermore the natural condition:

$$u(y,t) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty \quad t > 0 \quad (9)$$

has also to be satisfied. It is a consequence of the fact that the fluid is at rest at infinity and there is no shear in the free stream.

**Solution of the problem**

*Calculation of the velocity field*

First of all we solve the velocity field for sine oscillation. Applying the Laplace transform to eq. (4), using the Laplace transform formula for sequential fractional derivatives [39, 40] and taking into account the initial conditions (7)<sub>1,2</sub>, we find that:

$$\frac{\partial^2 \bar{u}(y, q)}{\partial y^2} - \frac{(1 + \lambda^\alpha q^\alpha)(q + M)}{\nu(1 + \lambda_r^\beta q^\beta)} \bar{u}(y, q) = 0 \tag{10}$$

The boundary conditions are:

$$\bar{u}(0, q) = \frac{U\omega}{q^2\omega^2} \quad \text{and} \quad \bar{u}(y, q) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty \tag{11}$$

where  $\bar{u}(y, q)$  is the Laplace transform of  $u(y, t)$  and  $q$  is a transform parameter. Solving eqs. (10) and (11), we get:

$$\bar{u}(y, q) = \frac{U\omega}{q^2\omega^2} \exp\left\{-\sqrt{\frac{(1 + \lambda^\alpha q^\alpha)(q + M)}{\nu(1 + \lambda_r^\beta q^\beta)}} y\right\} \tag{12}$$

In order to obtain  $u(y, t) = L^{-1}\{\bar{u}(y, q)\}$  and to avoid the lengthy and burdensome calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method [25-30]. However, for a suitable presentation of the velocity field, we firstly rewrite eq. (12) in series form:

$$\begin{aligned} \bar{u}(y, q) = & \frac{U\omega}{q^2 + \omega^2} + U\omega \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} \\ & \cdot \sum_{n=0}^{\infty} \frac{\Gamma\left(1 - \frac{k}{2}\right)\Gamma\left(m - \frac{k}{2}\right)\Gamma\left(n + \frac{k}{2}\right)(-\lambda_r^\beta)^n}{n!\Gamma\left(-\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}\right)} \frac{1}{q^{-\frac{k}{2} + l - \alpha m - \beta n + 2j + 2}} \end{aligned} \tag{13}$$

Inverting eq. (13) by means of discrete inverse Laplace transform, we find that:

$$\begin{aligned} u(y, t) = & U \sin(\omega t) + U\omega \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \\ & \cdot \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} t^{-\frac{k}{2} + l - \alpha m + 2j + 1} \sum_{n=0}^{\infty} \frac{\Gamma\left(1 - \frac{k}{2}\right)\Gamma\left(m - \frac{k}{2}\right)\Gamma\left(n + \frac{k}{2}\right)\left(-\frac{\lambda_r^\beta}{t^\beta}\right)^n}{n!\Gamma\left(-\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}\right)\Gamma\left(-\frac{k}{2} + l - \alpha m - \beta n + 2j + 2\right)} \end{aligned} \tag{14}$$

In terms of the Fox-H function [41], we write the above relation in a simpler form:

$$\begin{aligned} u_s(y, t) = & U \sin(\omega t) + U\omega \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} t^{-\frac{k}{2} + l - \alpha m + 2j + 1} \\ & \cdot H_{3,5}^{1,3} \left[ \begin{matrix} \lambda_r^\beta \\ t^\beta \end{matrix} \left| \begin{matrix} \left(1 - l + \frac{k}{2}, 0\right), \left(1 - m + \frac{k}{2}, 0\right), \left(1 - \frac{k}{2}, 1\right) \\ (0, 1) \left(1 + \frac{k}{2}, 0\right) \left(1 + \frac{k}{2}, 0\right) \left(1 - \frac{k}{2}, 0\right) \left(\frac{k}{2} - 1 + \alpha m - 2j - 1, \beta\right) \end{matrix} \right. \right] \end{aligned} \tag{15}$$

where the property of the Fox H-function is [41]:

$$\sum_{n=0}^{\infty} \frac{(-z)^n \prod_{j=1}^p \Gamma(\alpha_j + A_j n)}{n! \prod_{j=1}^q \Gamma(b_j + B_j n)} = H_{p,q+1}^{1,p} \left[ z \left| \begin{matrix} (1-a_1, A_1), \dots, (1-a_p, A_p) \\ (0,1)(1-b_1, B_1), \dots, (1-b_q, B_q) \end{matrix} \right. \right] \quad (16)$$

Proceeding in the same way as above, the velocity field corresponding to cosine oscillation is:

$$u_c(y, t) = H(t) \cos(\omega t) + UH(t) \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} t^{-\frac{k}{2} + l - \alpha m + 2j} \cdot \\ \cdot H_{3,5}^{1,3} \left[ \frac{\lambda_r^\beta}{t^\beta} \left| \begin{matrix} \left(1-l+\frac{k}{2}, 0\right), \left(1-m+\frac{k}{2}, 0\right), \left(1-\frac{k}{2}, 1\right) \\ (0,1) \left(1+\frac{k}{2}, 0\right) \left(1+\frac{k}{2}, 0\right) \left(1-\frac{k}{2}, 0\right) \left(\frac{k}{2}-1+\alpha m-2j-1, \beta\right) \end{matrix} \right. \right] \quad (17)$$

In order to justify the initial conditions (7)<sub>1,2</sub>, we use the initial value theorem of Laplace transform [42]. It is easy to see that the exact solutions (15) and (17) satisfy the boundary condition (8).

### Calculation of the shear stress

Applying the Laplace transform to eq. (5) and using the initial condition (7)<sub>3</sub>, we find that:

$$\bar{\tau}(y, q) = \mu \frac{1 + \lambda_r^\beta q^\beta}{1 + \lambda^\alpha q^\alpha} \frac{\partial \bar{u}(y, q)}{\partial y} \quad (18)$$

where  $\bar{\tau}(y, q)$  is the Laplace transform of  $\tau(y, t)$ . Introducing eq. (12) in (18), we find that:

$$\tau(y, q) = -\frac{\mu U \omega \sqrt{(1 + \lambda_r^\beta q^\beta)(q + M)}}{\sqrt{\nu}(q^2 + \omega^2) \sqrt{(1 + \lambda^\alpha q^\alpha)}} \exp \left\{ - \left[ \sqrt{\frac{1 + \lambda^\alpha q^\alpha (q + M)}{\nu(1 + \lambda_r^\beta q^\beta)}} \right] y \right\} \quad (19)$$

In order to obtain a more suitable form of  $\tau(y, t)$ , we rewrite eq. (19) in series form:

$$\bar{\tau}(y, q) = -\frac{\mu U \omega}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} \cdot \\ \cdot \sum_{n=0}^{\infty} \frac{\Gamma\left(1 - \frac{k+2}{2}\right) \Gamma\left(m - \frac{k-1}{2}\right) \Gamma\left(n + \frac{k-1}{2}\right) (-\lambda_r^\beta)^n}{n! \Gamma\left(-\frac{k+1}{2}\right) \Gamma\left(-\frac{k-1}{2}\right) \Gamma\left(\frac{k-1}{2}\right)} \frac{1}{q^{-\frac{k+1}{2} + l - \alpha m - \beta n + 2j + 2}} \quad (20)$$

Taking the Laplace inverse of eq. (20), we get:

$$\tau(y, t) = -\frac{\mu U \omega}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} t^{-\frac{k+1}{2} + l - \alpha m + 2j + 1} \cdot \\ \cdot \sum_{n=0}^{\infty} \frac{\Gamma\left(1 - \frac{k+1}{2}\right) \Gamma\left(m - \frac{k-1}{2}\right) \Gamma\left(n + \frac{k-1}{2}\right) \left(\frac{-\lambda_r^\beta}{t^\beta}\right)^n}{n! \Gamma\left(-\frac{k+1}{2}\right) \Gamma\left(-\frac{k-1}{2}\right) \Gamma\left(\frac{k-1}{2}\right) \Gamma\left(-\frac{k+1}{2} + l - \alpha m - \beta n + 2j + 2\right)} \quad (21)$$

or equivalently:

$$\tau_s(y, t) = -\frac{\mu U \omega}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} t^{-\frac{k+1}{2}+l-\alpha m+2j+1} \cdot H_{3,5}^{1,3} \left[ \frac{\lambda_r^\beta}{t^\beta} \left( \begin{matrix} \left(1-l+\frac{k+1}{2}, 0\right), \left(1-m+\frac{k-1}{2}, 0\right), \left(1-\frac{k-1}{2}, 1\right) \\ (0,1) \left(1+\frac{k+1}{2}, 0\right) \left(1+\frac{k-1}{2}, 0\right) \left(1-\frac{k-1}{2}, 0\right) \left(\frac{k+1}{2}-1+\alpha m-2j-1, \beta\right) \end{matrix} \right) \right] \quad (22)$$

The shear stress corresponding to cosine oscillation is:

$$\tau_c(y, t) = -\frac{\mu U \omega}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \sum_{m=0}^{\infty} \frac{(-\lambda^\alpha)^m}{m!} t^{-\frac{k+1}{2}+l-\alpha m+2j} \cdot H_{3,5}^{1,3} \left[ \frac{\lambda_r^\beta}{t^\beta} \left( \begin{matrix} \left(1-l+\frac{k+1}{2}, 0\right), \left(1-m+\frac{k-1}{2}, 0\right), \left(1-\frac{k-1}{2}, 1\right) \\ (0,1) \left(1+\frac{k+1}{2}, 0\right) \left(1+\frac{k-1}{2}, 0\right) \left(1-\frac{k-1}{2}, 0\right) \left(\frac{k+1}{2}-1+\alpha m-2j-1, \beta\right) \end{matrix} \right) \right] \quad (23)$$

**The special cases**

*Ordinary MHD Oldroyd-B fluid*

Making  $\alpha \rightarrow 1$  and  $\beta \rightarrow 1$  into eqs. (15), (17), (22), and (23), we obtain the velocity fields and shear stresses for ordinary MHD Oldroyd-B fluid.

*Fractional MHD Maxwell fluid*

Making  $\lambda_r \rightarrow 0$  in eqs. (14) and (21), we obtain the velocity field:

$$u_{SM}(y, t) = U \sin(\omega t) + U \omega \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} t^{-\frac{k}{2}+l+2j+1} \cdot H_{2,4}^{1,2} \left[ \frac{\lambda^\alpha}{t^\alpha} \left( \begin{matrix} \left(1-l+\frac{k}{2}, 0\right), \left(1+\frac{k}{2}, 1\right) \\ (0,1) \left(1+\frac{k}{2}, 0\right) \left(1+\frac{k}{2}, 0\right) \left(\frac{k}{2}-1-2j-1, \alpha\right) \end{matrix} \right) \right] \quad (24)$$

and the associate shear stress:

$$\tau_s(y, t) = -\frac{\mu U \omega}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} t^{-\frac{k+1}{2}+l+2j+1} \cdot H_{2,4}^{1,2} \left[ \frac{\lambda^\alpha}{t^\alpha} \left( \begin{matrix} \left(1-l+\frac{k+1}{2}, 0\right), \left(1+\frac{k-1}{2}, 0\right), \left(1-\frac{k-1}{2}, 1\right) \\ (0,1) \left(1+\frac{k+1}{2}, 0\right) \left(1+\frac{k-1}{2}, 0\right) \left(1-\frac{k+1}{2}-1-2j-1, \alpha\right) \end{matrix} \right) \right] \quad (25)$$

corresponding to the fractional MHD Maxwell fluid performing the same motion. The similar solutions for cosine oscillations of the boundary are:

$$u_{CM}(y, t) = UH(t) \cos(\omega t) + UH(t) \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} t^{-\frac{k}{2} + l + 2j} \cdot$$

$$\cdot H_{2,4}^{1,2} \left[ \frac{\lambda^\alpha}{t^\alpha} \left| \begin{matrix} \left(1-l + \frac{k}{2}, 0\right), \left(1 + \frac{k}{2}, 1\right) \\ (0,1) \left(1 + \frac{k}{2}, 0\right) \left(1 + \frac{k}{2}, 0\right) \left(\frac{k}{2} - 1 - 2j - 1, \alpha\right) \end{matrix} \right. \right] \quad (26)$$

$$\tau_{CM}(y, t) = -\frac{\mu UH(t)}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} t^{-\frac{k+1}{2} + l + 2j} \cdot$$

$$\cdot H_{2,4}^{1,2} \left[ \frac{\lambda^\alpha}{t^\alpha} \left| \begin{matrix} \left(1-l + \frac{k+1}{2}, 0\right), \left(1 + \frac{k-1}{2}, 1\right) \\ (0,1) \left(1 + \frac{k+1}{2}, 0\right) \left(1 + \frac{k-1}{2}, 0\right) \left(\frac{k+1}{2} - 1 - 2j - 1, \alpha\right) \end{matrix} \right. \right] \quad (27)$$

**Ordinary MHD Maxwell fluid**

Making  $\alpha \rightarrow 1$  in eqs. (24)-(27), we obtain the velocity fields and the shear stresses corresponding to MHD ordinary Maxwell fluid.

*Fractional MHD second grade fluid*

Making  $\lambda \rightarrow 0$  in eqs. (15), (17), (22), and (23), we obtain the velocity fields and the associated shear stresses:

$$u_{SSG}(y, t) = U \sin(\omega t) + U\omega \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} \cdot$$

$$\cdot t^{-\frac{k}{2} + l + 2j + 1} H_{2,4}^{1,2} \left[ \frac{\lambda_\rho^\beta}{t^\beta} \left| \begin{matrix} \left(1-l + \frac{k}{2}, 0\right), \left(1 - \frac{k}{2}, 1\right) \\ (0,1) \left(1 + \frac{k}{2}, 0\right) \left(1 - \frac{k}{2}, 0\right) \left(\frac{k}{2} - 1 - 2j - 1, \beta\right) \end{matrix} \right. \right] \quad (28)$$

$$\tau_{SSG}(y, t) = -\frac{\mu U\omega}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} t^{-\frac{k+1}{2} + l + 2j + 1} \cdot$$

$$\cdot H_{2,4}^{1,2} \left[ \frac{\lambda_\rho^\beta}{t^\beta} \left| \begin{matrix} \left(1-l + \frac{k+1}{2}, 0\right), \left(1 - \frac{k-1}{2}, 1\right) \\ (0,1) \left(1 + \frac{k+1}{2}, 0\right) \left(1 - \frac{k-1}{2}, 0\right) \left(\frac{k+1}{2} - 1 - 2j - 1, \beta\right) \end{matrix} \right. \right] \quad (29)$$

$$u_{CSG}(y, t) = UH(t) \cos(\omega t) + UH(t) \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{y}{\sqrt{\nu}} \right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} t^{-\frac{k}{2} + l + 2j} \cdot$$

$$\cdot H_{2,4}^{1,2} \left[ \frac{\lambda_r^\beta}{t^\beta} \left| \begin{matrix} \left(1-l + \frac{k}{2}, 0\right), \left(1 - \frac{k}{2}, 1\right) \\ (0,1) \left(1 + \frac{k}{2}, 0\right) \left(1 - \frac{k}{2}, 0\right) \left(\frac{k}{2} - 1 - 2j, \beta\right) \end{matrix} \right. \right] \quad (30)$$

$$\tau_{\text{CSG}}(y, t) = -\frac{\mu UH(t)}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k \sum_{l=0}^{\infty} \frac{(-M)^l}{l!} t^{-\frac{k+1}{2}+l+2j} \cdot$$

$$\cdot H_{2,4}^{1,2} \left[ \frac{\lambda_{\rho}^{\beta}}{t^{\beta}} \left| \begin{array}{c} \left(1-l+\frac{k+1}{2}, 0\right), \left(1-\frac{k-1}{2}, 1\right) \\ (0,1), \left(1+\frac{k+1}{2}, 0\right), \left(1-\frac{k-1}{2}, 0\right), \left(\frac{k+1}{2}-1-2j, \beta\right) \end{array} \right. \right] \quad (31)$$

corresponding to fractional MHD second grade fluid are obtained.

### Ordinary MHD second grade fluid

Making  $\beta \rightarrow 1$  into eqs. (28)-(31), we obtain the velocity field and corresponding shear stress for MHD ordinary second grade fluid.

#### MHD Newtonian fluid

Making  $\lambda$  and  $\lambda_r \rightarrow 0$  into eqs. (14) and (21), we obtain the solutions for velocity field and shear stress corresponding to sine oscillation for MHD Newtonian fluid:

$$u_{\text{SNG}}(y, t) = U \sin(\omega t) + U\omega \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k t^{-\frac{k}{2}+2j+1} \cdot$$

$$\cdot H_{1,3}^{1,1} \left[ Mt \left| \begin{array}{c} \left(1+\frac{k}{2}, 1\right) \\ (0,1), \left(1+\frac{k}{2}, 0\right), \left(\frac{k}{2}-2j-l, 1\right) \end{array} \right. \right] \quad (32)$$

$$\tau_{\text{SN}}(y, t) = -\frac{\mu U\omega}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k t^{-\frac{k+1}{2}+2j+1} \cdot$$

$$\cdot H_{1,3}^{1,1} \left[ Mt \left| \begin{array}{c} \left(1+\frac{k+1}{2}, 0\right) \\ (0,1), \left(1+\frac{k+1}{2}, 0\right), \left(\frac{k+1}{2}-2j-l, 1\right) \end{array} \right. \right] \quad (33)$$

The solutions for the cosine oscillations are given by:

$$u_{\text{CN}}(y, t) = UH(t) \cos(\omega t) + UH(t) \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k t^{-\frac{k}{2}+2j} \cdot$$

$$\cdot H_{1,3}^{1,1} \left[ Mt \left| \begin{array}{c} \left(1+\frac{k}{2}, 1\right) \\ (0,1), \left(1+\frac{k}{2}, 0\right), \left(\frac{k}{2}-2j, 1\right) \end{array} \right. \right] \quad (34)$$

$$\tau_{CN}(y, t) = -\frac{\mu UH(t)}{\sqrt{\nu}} \sum_{j=0}^{\infty} (-\omega^2)^j \sum_{k=1}^{\infty} \frac{1}{k!} \left(-\frac{y}{\sqrt{\nu}}\right)^k t^{-\frac{k+1}{2}+2j} \cdot {}_3H_{1,3}^{1,1} \left[ Mt \left| \begin{matrix} \left(1 + \frac{k+1}{2}, 1\right) \\ (0, 1), \left(1 + \frac{k+1}{2}, 0\right), \left(\frac{k+1}{2} - 2j, 1\right) \end{matrix} \right. \right] \quad (35)$$

Making  $M = \omega = 0$  into eqs. (34) and (35) we find that:

$$u_N(y, t) = UH_{0,2}^{1,0} \left[ \frac{y}{\sqrt{\nu t}} \left| (0, 1), \left(0, -\frac{1}{2}\right) \right. \right], \quad \tau_N(y, t) = -\frac{\mu U}{\sqrt{\nu t}} H_{0,2}^{1,0} \left[ \frac{y}{\sqrt{\nu t}} \left| (0, 1), \left(\frac{3}{2}, -\frac{1}{2}\right) \right. \right] \quad (36)$$

Using the definition of Fox  $H$ -function and the series expression of the error function, we can easily prove that:

$$H_{0,2}^{1,0} \left[ \frac{y}{\sqrt{\nu t}} \left| (0, 1), \left(0, -\frac{1}{2}\right) \right. \right] = \operatorname{erfc} \left( \frac{z}{2} \right), \quad H_{0,2}^{1,0} \left[ \frac{y}{\sqrt{\nu t}} \left| (0, 1), \left(\frac{3}{2}, -\frac{1}{2}\right) \right. \right] = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{z^2}{4} \right) \quad (37)$$

Substituting eq. (37) into eq. (36), we recover the classical solutions:

$$u_N(y, t) = U \operatorname{erfc} \left( \frac{y}{2\sqrt{\nu t}} \right), \quad \tau_N(y, t) = -\frac{\mu U}{\sqrt{\pi \nu t}} \exp \left( -\frac{y^2}{2\nu t} \right) \quad (38)$$

corresponding to the first problem of Stokes.

### Hydrodynamic fluids

Finally, making  $M \rightarrow 0$  into eqs. (15), (17), (22), and (23), the solutions for hydrodynamic fluid are obtained.

### Numerical results and discussions

In the previous sections, we have established exact analytical solutions for a flow problem of fractional MHD Oldroyd-B fluids. In order to capture some relevant physical aspects of the obtained results, several graphs are depicted in this section. The numerical results illustrate the velocity and the shear stress profiles for the flow induced by a rigid oscillating plate. We interpret these results with respect to the variations of emerging parameters of interest.

Figures 1 are sketched to show the velocity  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  profiles at different values of  $y$ . From these figures, it can be seen that the amplitude of the fluid oscillation decays away from the plate and approaches to zero. Figures 2 and 3 are prepared to show the effect of time  $t$  and frequency of oscillation  $\omega$  on velocity and shear stress profiles. It is clear that the amplitude of fluid oscillations decreases when time or frequency of oscillation  $\omega$  increases. However, it is noticeable that this result cannot be generalized for all time because it possesses monotonic behavior. The effect of magnetic parameter  $M$  and kinematic viscosity  $\nu$  is shown in figs. 4 and 5. The increasing values of these parameters also decrease the amplitude of fluid oscillation. However, the shear stress for both cases change its monotonicity away from the plate. The influence of relaxation and retardation times  $\lambda$  and  $\lambda_r$  is depicted in figs. 6 and 7. As ex-

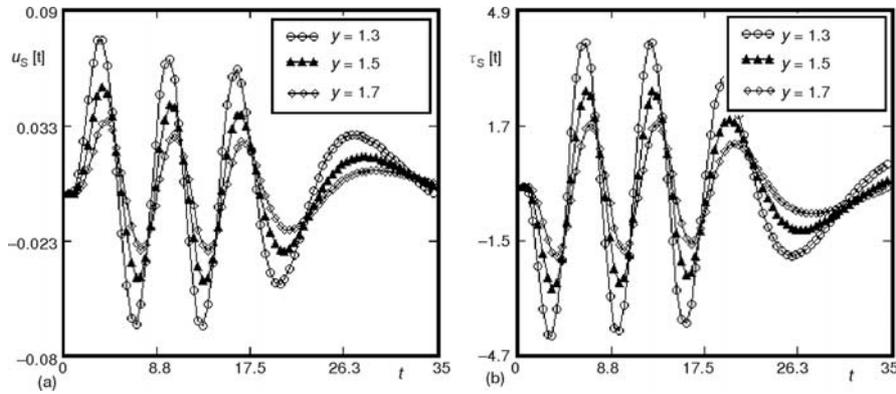


Figure 1. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, \lambda = 2, \lambda_r = 1.5, M = 0.5, \alpha = 0.5, \beta = 0.4$ , and different values of  $y$

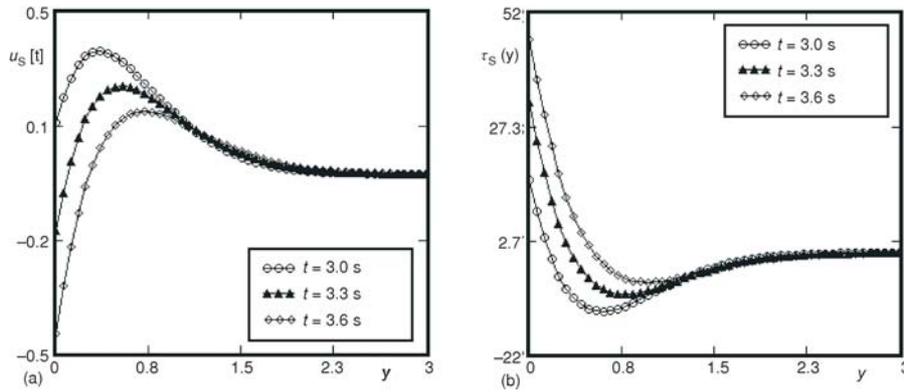


Figure 2. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, \lambda = 2, \lambda_r = 1.5, M = 0.5, \alpha = 0.5, \beta = 0.2$ , and different values of  $t$

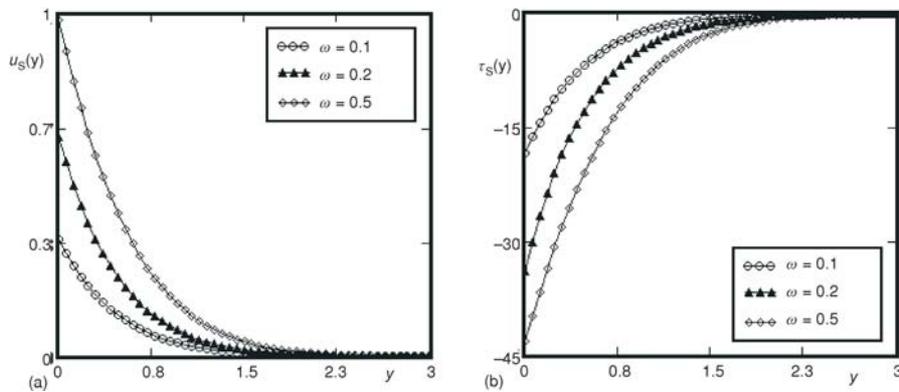


Figure 3. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \nu = 0.186, \mu = 26, \lambda = 2, \lambda_r = 1.5, M = 0.5, \alpha = 0.5, \beta = 0.4, t = 3.5$  s, and different values of  $\omega$

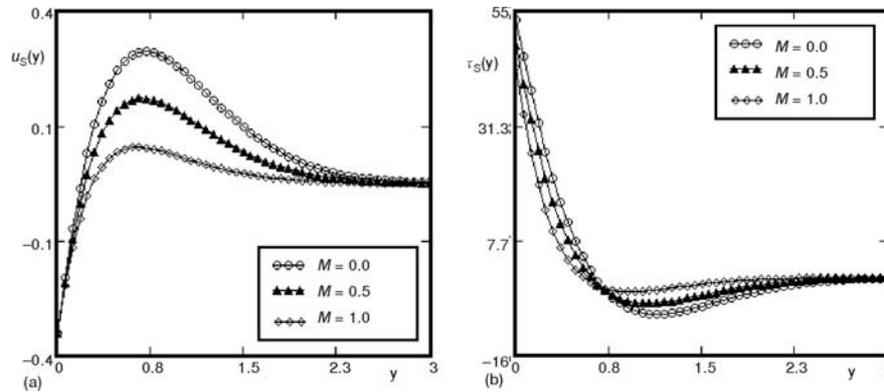


Figure 4. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, \lambda = 2, \lambda_r = 1.5, \alpha = 0.5, \beta = 0.4, t = 3.5$  s, and different values of  $M$

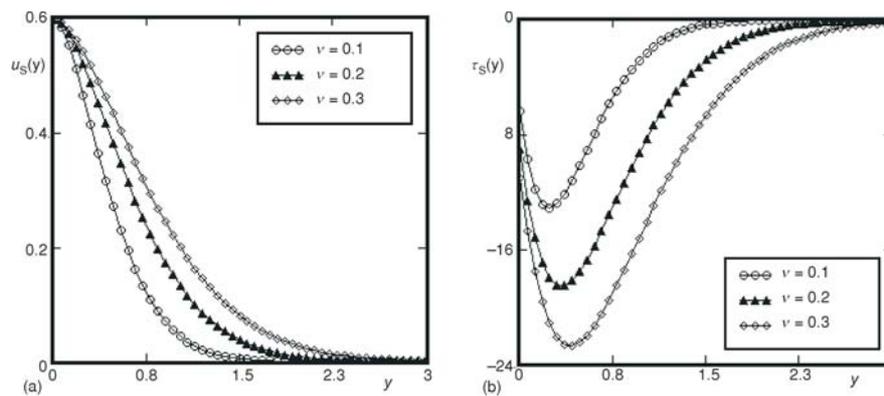


Figure 5. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \rho = 140, \lambda = 2, \lambda_r = 1.5, \alpha = 0.5, \beta = 0.4, t = 3.5$  s, and different values of  $\nu$

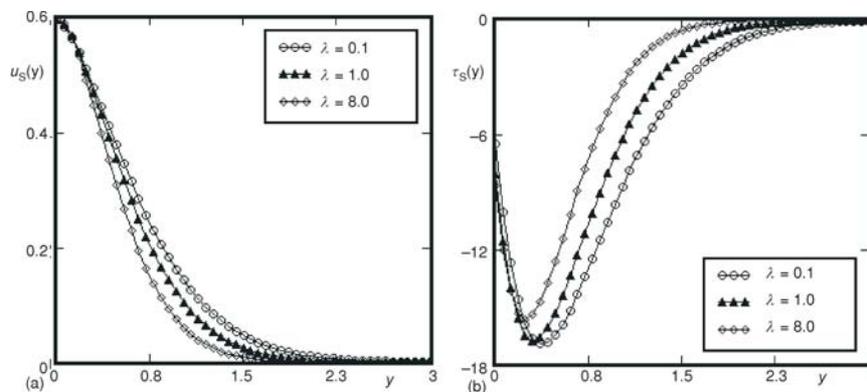


Figure 6. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, \lambda_r = 0.1, M = 0.5, \alpha = 0.3, \beta = 0.2, t = 3.5$  s, and different values of  $\lambda$

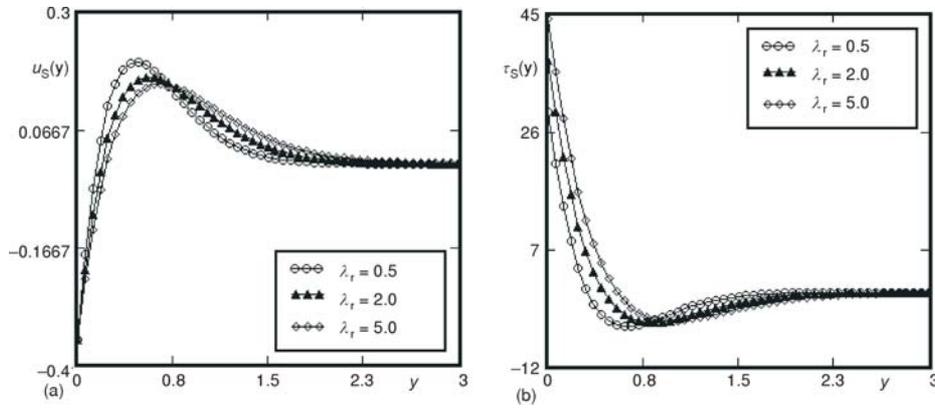


Figure 7. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, \lambda = 5, M = 0.5, \alpha = 0.5, \beta = 0.4, t = 3.5$  s, and different values of  $\lambda_r$ .

pected their effects on fluid oscillations are opposite. For instance, it is easy to see that for large values of  $y$ , the behavior of relaxation time is shear-thickening while retardation time has shear-thinning behavior on fluid oscillations.

More important for us is to discuss the effects of fractional parameters  $\alpha$  and  $\beta$  of the model. In figs. 8 and 9, we depict the profiles of velocity  $u_s(y, t)$  and shear stress  $\tau_s(y, t)$  for three different values of  $\alpha$  and  $\beta$ . It is observed from these figures that  $\alpha$  shows a shear-thinning behavior while  $\beta$  depicts a shear-thickening behavior. As  $\beta$  strengthens the shear-thickening, the amplitude of fluid oscillation away from the plate is also reduced while  $\alpha$  depicts an opposite behavior. Finally, for comparison, the velocity field and the shear stress corresponding to the four models (fractional Oldroyd-B, fractional Maxwell, fractional second grade and Newtonian) for magnetic and without magnetic effect are together depicted in figs. 10 and 11. It is clearly seen from these figures that fractional Maxwell fluids have largest and the fractional M0nd grade fluids have the smallest amplitude of fluid oscillations for velocity field as well as shear stress, whether magnetic effect is present or not. The units of the material parameters in all figures are SI units.

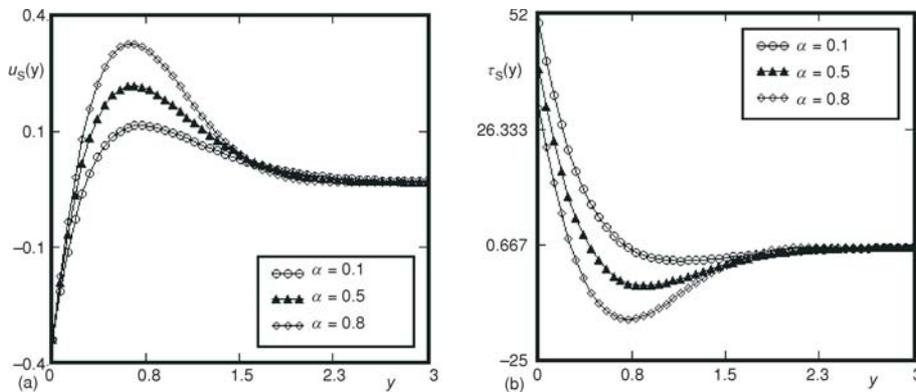


Figure 8. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, \lambda = 2, \lambda_r = 1.5, M = 0.5, \beta = 0.1, t = 3.5$  s, and different values of  $\alpha$ .

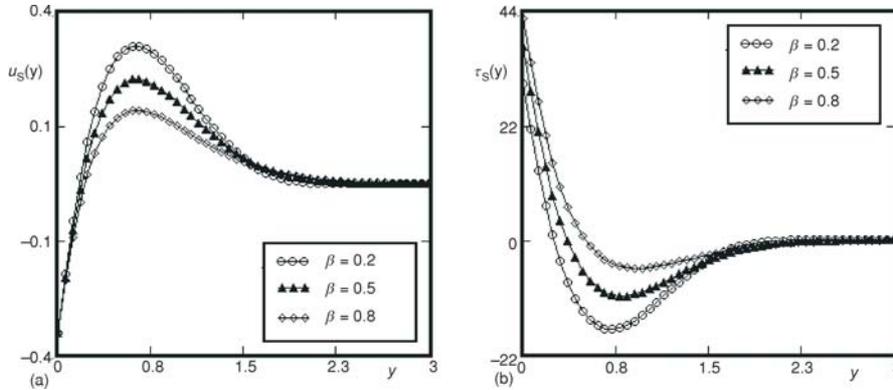


Figure 9. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  given by eqs. (15) and (22), for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, \lambda = 2, \lambda_r = 1.5, M = 0.5, \alpha = 0.9, t = 3.5$  s, and different values of  $\beta$

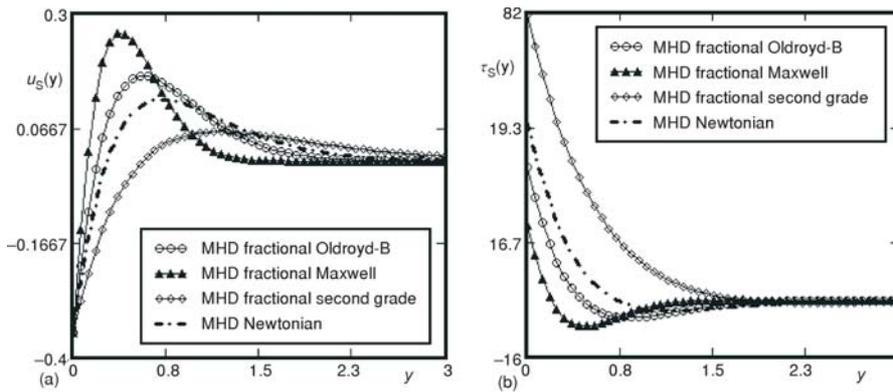


Figure 10. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  for fractional MHD Oldroyd-B ( $\lambda = 5, \lambda_r = 1.5$ ), fractional MHD Maxwell ( $\lambda = 5, \lambda_r = 1.5$ ), fractional MHD second grade ( $\lambda = 0, \lambda_r = 1.5$ ), and MHD Newtonian ( $\lambda = 0, \lambda_r = 0$ ) fluids, for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, M = 0.5, \alpha = 0.5, \beta = 0.4$ , and  $t = 3.5$  s

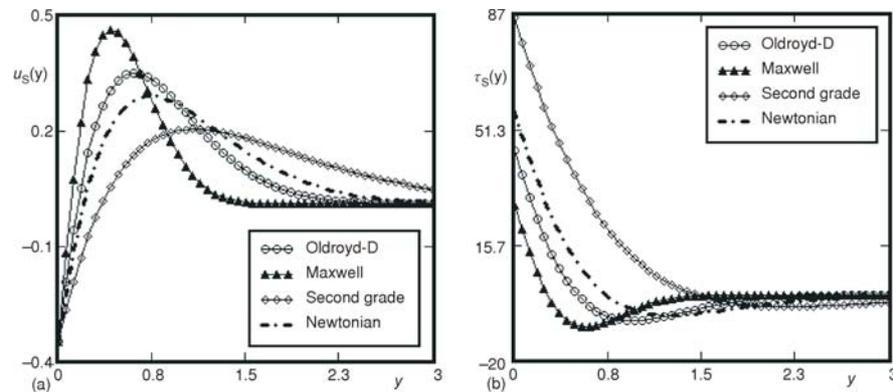


Figure 11. Profiles of the velocity field  $u_s(y, t)$  and the shear stress  $\tau_s(y, t)$  for fractional MHD Oldroyd-B ( $\lambda = 5, \lambda_r = 1.5$ ), fractional MHD Maxwell ( $\lambda = 5, \lambda_r = 0$ ), fractional MHD second grade ( $\lambda = 0, \lambda_r = 1.5$ ), and MHD Newtonian ( $\lambda = 0, \lambda_r = 0$ ) fluids, for  $U = 1, \omega = 1, \nu = 0.186, \mu = 26, M = 0.5, \alpha = 0.5, \beta = 0.4$ , and  $t = 3.5$  s

### Concluding remarks

The purpose of this paper is to establish exact solutions corresponding to oscillating motion of a MHD Oldroyd-B fluid with fractional derivatives. Analytical expressions for velocity fields and the corresponding shear stresses for flows due to oscillations of an infinite flat plate were determined using discrete Laplace transform for sequential fractional derivatives. The solutions that have been obtained, presented in series form in terms of the Fox H-functions, satisfy all imposed initial and boundary conditions. In special cases the solutions for ordinary MHD Oldroyd-B, fractional and ordinary MHD Maxwell, fractional and ordinary MHD second grade, and MHD Newtonian fluid as well as those for hydrodynamic fluids are obtained from general solutions. Many previously known results are recovered from the present results, such as Stokes' first and second problems for MHD and hydrodynamic Oldroyd-B, Maxwell, second grade and Newtonian fluids. The results categorically indicate the following findings.

- The amplitude of the fluid oscillation decays away from the plate and approaches to zero.
- The amplitude of fluid oscillation decreases when time or frequency of oscillation  $\omega$  increases, however this result cannot be generalized.
- The influence of magnetic field  $M$  and kinematic viscosity  $\nu$  decrease the amplitude of oscillation.
- The relaxation time implies shear-thickening while the retardation time implies shear-thinning behavior on the fluid motion.
- The fractional Maxwell fluid have the largest and fractional second grade have the smallest amplitude of oscillations independent of the magnetic field.

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Paper submitted: July 31, 2011

Paper revised: October 22, 2011

Paper accepted: October 26, 2011