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RESEARCH ARTICLE - MECHANICAL ENGINEERING

Starting Solutions for Motion of a Maxwell Fluid Over an Infinite Plate that Applies an Oscillating Shear to the Fluid

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Abstract Unsteady motion of a Maxwell fluid over an infinite plate that applies an oscillating shear to the fluid is studied by means of integral transforms. The obtained solutions satisfy all initial and boundary conditions. They are presented as a sum of steady-state and transient solutions and can easily be reduced to the similar solutions for Newtonian fluids. They describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the motion of the fluid is described by the steady-state solutions which are periodic in time and independent of the initial conditions. However, they satisfy the governing equations and boundary conditions. Finally, the required time to reach the steady-state, as well as a comparison between models, is determined by graphical illustrations.

Keywords Starting solutions · Maxwell fluid · Oscillating shear stress · Infinite plate

الخلاصة

تمت دراسة الحركة المتقلبة لسائل ماكسويل فوق لوحة غير محدودة يطبق قصّاً متأرجحاً للسائل عن طريق التحويلات التكاملية. والحلول التي تم الحصول عليها تلبي جميع الشروط الأولية والحدود، حيث قُدّمت كمجموع من حلول الحالة الثابتة والعابرة، ويمكن بسهولة أن تُخفّض إلى حلول مماثلة لسوائل نيوتن، وهي تصف حركة السائل لبعض الوقت بعد الشروع في ذلك. وبعد ذلك الوقت، وعندما يختفي العابرون تم وصف حركة السائل عن طريق حلول الحالة الثابتة التي هي دورية في الوقت، ومستقلة عن الظروف الأولية. ومع ذلك فإنها تُرضي المعادلات التي تحكم وشروط الحدود. وأخيرا تم تحديد الوقت المطلوب للوصول إلى الحالة الثابتة، وكذلك المقارنة بين النماذج عن طريق الرسوم البيانية التوضيحية.

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1 Introduction

There are many fluids with complex microstructure whose behavior cannot be described by the Navier–Stokes equations. These fluids, while exhibit a non-linear relationship between the stress and the rate of strain are called non-Newtonian fluids. Their motion has been an important subject in the field of chemical, biomedical, environmental engineering and science. Numerous models have been proposed to describe their behavior in different circumstances. They are usually classified as fluids of differential type, rate type and integral type. The rate type and differential models are used to describe the response of fluids that have slight memory such as dilute polymeric solutions. The first Vis-co-elastic rate type model, which is still being used widely in the literature, was given by Maxwell [1]. The rate type fluids are increasingly, being recognized as more appropriate in the modern technological applications. The governing equations for motions of these fluids lead to flow problems in which the order of differential equations exceeds the number of available conditions and their solutions are generally more difficult to be obtained.

The motion of a fluid over an infinite plate is of interest both for academic research and its practical applications. It can be obtained as a result of several effects such as various types of motion of boundaries or application of a shear stress on the wall. The flow of a fluid caused by the sinusoidal/co-sinusoidal oscillations of an infinite plate is called the Stokes' second problem [2], and has been successfully studied in [3–9]. The Stokes's second problem is considered in the assumption that the fluid adheres to the solid boundary (non-slip boundary condition), therefore, the velocity of the fluid on the plate is identical with the velocity of the plate. The non-slip boundary condition is one of the central tenets of the Navier–Stokes theory. More experiments are in favor of the non-slip boundary condition for a large class of flows. However, there are situations where the assumption of non-slip is not fulfilled. Navier [10] had proposed a slip boundary condition where the slip velocity depends linearly on the shear stress. In general, the slip velocity strongly depends on the shear stress, and most governing equations developed for slip assume that it depends only on the shear stress. The slip that appears at the wall has led to the study of an interesting class of problems in which the shear stress (or the shear rate) is given on the solid boundary. The unsteady motion of a viscous fluid between two side walls perpendicular to a plate that applies an oscillating shear stress to the fluid was studied in [11]. More useful results regarding steady flows of Newtonian or Maxwell fluids induced by an oscillating plate can be found in the references [12–15]. Other interesting results about flows with slip or non-slip conditions are in the references [16–19].

The aim of this paper was to provide exact solutions for the motion of a Maxwell fluid over an infinite plate under assumption that the shear rate is given on the plate. More exactly, we consider the plate situated in the plane $y = 0$ of a Cartesian coordinate system Oxyz, the domain of the flow is the half-space $y > 0$ and the shear rate is given on the plate by the expression $\frac{\partial V(y,t)}{\partial y}|_{y=0} = \frac{f}{\mu}\sin(\omega t)$ or $\frac{\partial V(y,t)}{\partial y}|_{y=0} = \frac{f}{\mu}\cos(\omega t)$. The solutions of the initial-boundary value problems that govern the flow are obtained by means of the integral transforms method. It is important to point out that these solutions are written as a sum between steady-state solutions (permanent solutions) and transient solutions. Closed-form exact solutions for this type of the flow are new in the literature and can easily be reduced to the similar solutions for Newtonian fluids. The required time after which the transient solution can be neglected (therefore, the fluid flows according to the permanent solution) is determined by graphical illustrations. The influence of the frequency and the relaxation time on the motion is also studied.

2 Governing Equations

The constitutive equations for an incompressible Maxwell fluid are given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \ \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu\mathbf{A}, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A}$ is the first Rivlin–Ericksen tensor, $-p$I denotes the indeterminate spherical stress, μ is the dynamic viscosity, λ is the relaxation time, the superscript T indicates the transpose operation and the dot denotes the material time differentiation. In the following, we shall look for a velocity field of the form

$$\mathbf{V} = \mathbf{V}(y,\ t) = \mathbf{V}(y,\ t)\mathbf{i}, \tag{2}$$

where $\mathbf{i}$ denotes the unit vector along the x direction of the Cartesian coordinate system Oxyz. For such flows, the constraint of incompressibility is automatically satisfied. We also assume that the extra-stress $\mathbf{S}$ depends only on y and t. By substituting (1) and (2) into the balance of linear momentum equation, neglecting body forces and assuming that there is no applied pressure gradient along the x axis, we attain to the following partial differential equations

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t) = \mu\frac{\partial V(y,t)}{\partial y};\quad y,t>0, \tag{3}$$

$$\lambda\frac{\partial^2 V(y,t)}{\partial t^2} + \frac{\partial V(y,t)}{\partial t} = \nu\frac{\partial^2 V(y,t)}{\partial y^2};\ \ y,t>0, \tag{4}$$

that govern the motion of the fluid. Here, $\nu = \mu/\rho$ is the kinematic viscosity of the fluid, ρ being its constant density and $\tau(y,t) = S_{xy}(y,t)$ is the non-trivial shear stress.

3 Statement of the Problem and Solutions

Let us consider an incompressible Maxwell fluid at rest over an infinite flat plate. After time $t = 0$, the plate applies an oscillating shear to the fluid $\frac{f}{\mu}\sin(\omega t)$ or $\frac{f}{\mu}\cos(\omega t)$ where f and ω are constants. Owing to the shear, the fluid is gradually moved. The governing equations are given by Eqs. (3) and (4) and the appropriate initial and boundary conditions are

$$V(y,0) = \frac{\partial V(y,0)}{\partial t} = 0,\quad \tau(y,0) = 0;\ \ y>0, \tag{5}$$

$$\frac{\partial V(y,t)}{\partial y}|_{y=0} = \frac{f}{\mu}\sin(\omega t),\quad \text{or}\quad \frac{f}{\mu}\cos(\omega t),\ \ t>0, \tag{6}$$

$$V(y,t)\to 0,\quad \text{as}\ \ y\to\infty. \tag{7}$$

3.1 Calculation of the Velocity Field

To determine the solution of the initial-boundary value problem (4)–(7), we shall use the Fourier cosine and Laplace transforms. All calculi will be presented for the sine oscillations only.

Multiplying Eq. (4) by $\sqrt{\frac{2}{\pi}}\cos(y\xi)$, integrating the result with respect to y from 0 to infinity and using Eqs. (5)–(7), we find that

$$\lambda\frac{\partial^2 V_c(\xi,t)}{\partial t^2} + \frac{\partial V_c(\xi,t)}{\partial t} + \nu\xi^2 V_c(\xi,t) = -\sqrt{\frac{2}{\pi}}\frac{f\sin(\omega t)}{\rho};\quad \xi,t>0, \tag{8}$$

$$V_c(\xi,0) = \frac{\partial V_c(\xi,0)}{\partial t} = 0\ \ \text{for}\ \xi>0, \tag{9}$$

where $V_c(\xi, t)$ is the Fourier cosine transform of the function $V(y, t)$. Applying the Laplace transform to Eq. (8) and using Eq. (9), we have

$$\overline{V_c}(\xi, q) = -\sqrt{\frac{2}{\pi}}\frac{f}{\rho}\frac{\omega}{q^2+\omega^2}\frac{1}{\lambda q^2+q+\nu\xi^2}, \tag{10}$$

where $\overline{V}_c(\xi, q)$ denotes the Laplace transform of the function $V_c(\xi, t)$.

Equation (10) can be written as a sum, namely

$$\overline{V}_c(\xi, q) = \overline{V}_{c1}(\xi, q) + \overline{V}_{c2}(\xi, q) + \overline{V}_{c3}(\xi, q), \tag{11}$$

where

$$\begin{aligned}
\overline{V}_{c1}(\xi, q) &= \sqrt{\frac{2}{\pi}}\frac{f\omega}{\rho}\frac{1}{(\nu\xi^2-\lambda\omega^2)^2+\omega^2}\frac{q}{q^2+\omega^2},\\
\overline{V}_{c2}(\xi, q) &= -\sqrt{\frac{2}{\pi}}\frac{f}{\rho}\frac{(\nu\xi^2-\lambda\omega^2)}{(\nu\xi^2-\lambda\omega^2)^2+\omega^2}\frac{\omega}{q^2+\omega^2},\\
\overline{V}_{c3}(\xi, q) &= -\sqrt{\frac{2}{\pi}}\frac{f\omega}{\rho}\frac{1}{(\nu\xi^2-\lambda\omega^2)^2+\omega^2}\\
&\quad\times\left[\frac{\lambda q+1-\lambda(\nu\xi^2-\lambda\omega^2)}{\lambda q^2+q+\nu\xi^2}\right].
\end{aligned} \tag{12}$$

Applying the inverse Laplace transform and then the inverse Fourier cosine transform to Eq. $(12)_1$, we obtain

$$V_1(y, t) = \frac{2}{\pi}\frac{f\omega}{\mu\nu}\cos(\omega t)\int_0^\infty \frac{\cos(y\xi)}{\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2}\mathrm{d}\xi. \tag{13}$$

In view of Eq. (A_1) from Appendix, it results that

$$V_1(y, t) = \frac{f}{\mu}\cos(\omega t)\frac{e^{-yB}}{A^2+B^2}\left[A\cos(yA)+B\sin(yA)\right], \tag{14}$$

where

$$\begin{aligned}
A^2 &= \frac{\omega}{2\nu}\left[\sqrt{1+(\lambda\omega)^2}+(\lambda\omega)\right],\\
B^2 &= \frac{\omega}{2\nu}\left[\sqrt{1+(\lambda\omega)^2}-(\lambda\omega)\right].
\end{aligned} \tag{15}$$

Similarly, from Eq. $(12)_2$, we get

$$V_2(y, t) = -\frac{2}{\pi}\frac{f\omega}{\mu\nu}\sin(\omega t)\int_0^\infty \frac{\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)\cos(y\xi)}{\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2}\mathrm{d}\xi \tag{16}$$

which, in view of Eq. (A_2) from Appendix, becomes

$$V_2(y, t) = \frac{f}{\mu}\sin(\omega t)\frac{e^{-yB}}{A^2+B^2}\left[A\sin(yA)-B\cos(yA)\right]. \tag{17}$$

Direct computations lead to the following expression for $V_1(y, t) + V_2(y, t)$:

$$V_1(y, t)+V_2(y, t) = \frac{f}{\mu}\frac{e^{-yB}}{\sqrt{A^2+B^2}}\sin\left(\omega t-yA+\varphi+\frac{\pi}{2}\right), \tag{18}$$

where $\tan\varphi = B/A$.

To determine the inverse Laplace transform of $\overline{V}_{c3}(\xi, q)$, we write the function

$$\begin{aligned}
F(\xi, q) &= \left[\frac{\lambda q+1-\lambda(\nu\xi^2-\lambda\omega^2)}{\lambda q^2+q+\nu\xi^2}\right]\\
&= \frac{\lambda q+1-\lambda(\nu\xi^2-\lambda\omega^2)}{\lambda\left[\left(q+\frac{1}{2\lambda}\right)^2-\frac{1-4\lambda\nu\xi^2}{4\lambda^2}\right]}
\end{aligned} \tag{19}$$

in the following equivalent form:

$$\begin{aligned}
F(\xi, q) &= \frac{\left(q+\frac{1}{2\lambda}\right)}{\left(q+\frac{1}{2\lambda}\right)^2-\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}\right)^2}\\
&\quad+\frac{1-2\lambda(\nu\xi^2-\lambda\omega^2)}{\sqrt{1-4\lambda\nu\xi^2}}\frac{\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}}{\left(q+\frac{1}{2\lambda}\right)^2-\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}\right)^2}.
\end{aligned} \tag{20}$$

Applying the inverse Laplace transform and then the inverse Fourier cosine transform to Eq. $(12)_3$ and using Eq. (20), we obtain, for $V_3(y, t)$, the expression

$$\begin{aligned}
V_3(y, t) &= -\frac{2}{\pi}\frac{f\omega}{\mu\nu}e^{-\frac{t}{2\lambda}}\int_0^\infty \frac{\cos(y\xi)}{\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2}\\
&\quad\times\left[ch\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)+\frac{1-2\lambda(\nu\xi^2-\lambda\omega^2)}{\sqrt{1-4\lambda\nu\xi^2}}\right.\\
&\quad\times\left. sh\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)\right]\mathrm{d}\xi.
\end{aligned} \tag{21}$$

Finally, the velocity field corresponding to the sine oscillation of the shear is given by

$$\begin{aligned}
V_s(y, t) &= \frac{f}{\mu}\frac{e^{-yB}}{\sqrt{A^2+B^2}}\sin\left(\omega t-yA+\phi+\frac{\pi}{2}\right)\\
&\quad-\frac{2}{\pi}\frac{f\omega}{\mu\nu}e^{-\frac{t}{2\lambda}}\int_0^\infty \frac{\cos(y\xi)}{\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2}\\
&\quad\times\left[ch\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)+\frac{1-2\lambda(\nu\xi^2-\lambda\omega^2)}{\sqrt{1-4\lambda\nu\xi^2}}\right.\\
&\quad\times\left. sh\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)\right]\mathrm{d}\xi.
\end{aligned} \tag{22}$$

In a similar way, we obtain the velocity field

$$V_c(y,t) = \frac{f}{\mu}\frac{e^{-yB}}{\sqrt{A^2+B^2}}\cos\left(\omega t - yA + \phi + \frac{\pi}{2}\right) + \frac{2}{\pi}\frac{f}{\mu}e^{-\frac{t}{2\lambda}}\int_0^\infty \frac{\cos(y\xi)}{\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2} \times\left[(\xi^2-\lambda\omega^2/\nu)ch\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right) + \frac{\xi^2+\lambda\omega^2/\nu}{\sqrt{1-4\lambda\nu\xi^2}}sh\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)\right]d\xi. \quad (23)$$

corresponding to the cosine oscillations of the shear stress. The starting solutions (22) and (23) are presented as a sum between the steady-state and transient solutions. They describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the motion of the fluid is described by the steady-state solutions

$$V_{ss}(y,t) = \frac{f}{\mu}\frac{e^{-yB}}{\sqrt{A^2+B^2}}\sin\left(\omega t - yA + \varphi + \frac{\pi}{2}\right), \quad (24)$$

$$V_{cs}(y,t) = \frac{f}{\mu}\frac{e^{-yB}}{\sqrt{A^2+B^2}}\cos\left(\omega t - yA + \varphi + \frac{\pi}{2}\right) = \frac{f}{\mu}\frac{e^{-yB}}{\sqrt{A^2+B^2}}\sin(\omega t - yA + \varphi + \pi), \quad (25)$$

which are periodic in time and independent of initial conditions. However, they satisfy the governing equations and boundary conditions. Furthermore, as it was to be expected, they differ by a phase shift.

3.2 Calculation of the Shear Stress

Applying the Laplace transform to Eq. (3) and having in mind the initial condition $(5)_3$, we find that

$$\overline{\tau}(y,q) = \frac{\mu}{\lambda q+1}\frac{\partial\overline{V}(y,q)}{\partial y}; \quad y>0.$$

The function $\overline{V}(y,q)$ can immediately be obtained by applying the inverse Fourier cosine transform to Eq. (10). By combining these results, we find that

$$\overline{\tau}(y,q) = \frac{2f\omega\nu}{\pi}\int_0^\infty \xi\sin(y\xi)\frac{1}{(q^2+\omega^2)}\cdot\frac{1}{(1+\lambda q)} \cdot\frac{1}{(\lambda q^2+q+\nu\xi^2)}d\xi. \quad (26)$$

Equation (26) can also be written as a sum, namely

$$\overline{\tau}(y,q) = \overline{\tau}_1(y,q)+\overline{\tau}_2(y,q)+\overline{\tau}_3(y,q)+\overline{\tau}_4(y,q), \quad (27)$$

where

$$\overline{\tau}_1(y,q) = -\frac{2f\omega\nu}{\pi}\cdot\frac{q}{q^2+\omega^2} \times\int_0^\infty \frac{\xi\sin(y\xi)\left[1+\lambda\left(\nu\xi^2-\lambda\omega^2\right)\right]}{(1+\lambda^2\omega^2)\left[\left(\nu\xi^2-\lambda\omega^2\right)^2+\omega^2\right]}d\xi,$$

$$\overline{\tau}_2(y,q) = \frac{2f\nu}{\pi}\cdot\frac{\omega}{q^2+\omega^2} \times\int_0^\infty \frac{\xi\sin(y\xi)\left[\left(\nu\xi^2-\lambda\omega^2\right)-\lambda\omega^2\right]}{(1+\lambda^2\omega^2)\left[(\nu\xi^2-\lambda\omega^2)^2+\omega^2\right]}d\xi,$$

$$\overline{\tau}_3(y,q) = \frac{2f\omega\nu}{\pi}\cdot\frac{1}{\lambda q+1} \times\int_0^\infty \xi\sin(y\xi)\frac{\lambda^2}{\nu\xi^2(1+\lambda^2\omega^2)}d\xi,$$

$$\overline{\tau}_4(y,q) = \frac{2f\omega\nu}{\pi}\int_0^\infty \frac{\xi\sin(y\xi)}{\left[\left(\nu\xi^2-\lambda\omega^2\right)^2+\omega^2\right]\nu\xi^2} \cdot\frac{\lambda\left(\nu\xi^2-\lambda\omega^2\right)q+\nu\xi^2}{\lambda q^2+q+\nu\xi^2}d\xi. \quad (28)$$

Applying the inverse Laplace transform to Eqs. $(28)_{1,2}$ and using Eqs. (A_1–A_5) from Appendix, we find the expressions

$$\tau_1(y,t) = \frac{-f}{1+\lambda^2\omega^2}\cos(\omega t)\sin(Ay)e^{-By} - \frac{f\lambda\omega}{1+\lambda^2\omega^2}\cos(\omega t)\cos(Ay)e^{-By}. \quad (29)$$

$$\tau_2(y,t) = \frac{f}{1+\lambda^2\omega^2}\sin(\omega t)\cos(Ay)e^{-By} - \frac{f\lambda\omega}{1+\lambda^2\omega^2}\sin(\omega t)\sin(Ay)e^{-By}. \quad (30)$$

Lengthy but straightforward computation leads to the simple relation

$$\tau_1(y,t)+\tau_2(y,t) = \frac{fe^{-By}}{\sqrt{1+\lambda^2\omega^2}}\sin(\omega t - Ay - \psi) \quad (31)$$

where $\tan(\psi) = \lambda\omega$.

Applying the inverse Laplace transform to Eq. $(28)_3$ and using Eq. (A_3), we find that

$$\tau_3(y,t) = \frac{f\lambda\omega}{(1+\lambda^2\omega^2)}e^{-\frac{t}{\lambda}}, \quad (32)$$

The last factor of Eq. $(28)_4$, namely

$$G(y,q) = \frac{\lambda(\nu\xi^2-\lambda\omega^2)q+\nu\xi^2}{\lambda q^2+q+\nu\xi^2},$$

can be written in the equivalent form;

$$G(y,q)=(\nu\xi^2-\lambda\omega^2)\cdot\frac{(q+\frac{1}{2\lambda})}{(q+\frac{1}{2\lambda})^2-\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}\right)^2}+\frac{(\nu\xi^2+\lambda\omega^2)}{\sqrt{1-4\lambda\nu\xi^2}}\frac{\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}}{(q+\frac{1}{2\lambda})^2-\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}\right)^2}. \tag{33}$$

Applying the inverse Laplace transform to Eq. (28)$_4$ and using Eq. (33) we obtain

$$\tau_4(y,t)=\frac{2f}{\pi}\frac{\omega}{\nu}e^{-\frac{t}{2\lambda}}\int_0^\infty\frac{\sin(y\xi)}{\xi\left[\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2\right]}\times\left[(\xi^2-\lambda\omega^2/\nu)ch\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)+\frac{\xi^2+\lambda\omega^2/\nu}{\sqrt{1-4\lambda\nu\xi^2}}sh\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)\right]d\xi. \tag{34}$$

In view of Eqs. (31), (32) and (34), the final expression for the shear stress is

$$\tau_s(y,t)=\frac{fe^{-By}}{\sqrt{1+\lambda^2\omega^2}}\sin(\omega t-yA-\psi)+\frac{f\lambda\omega}{1+\lambda^2\omega^2}e^{-\frac{t}{\lambda}}+\frac{2f}{\pi}\frac{\omega}{\nu}e^{-\frac{t}{2\lambda}}\int_0^\infty\frac{\sin(y\xi)}{\xi\left[\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2\right]}\times\left[\left(\xi^2-\lambda\omega^2/\nu\right)ch\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)+\frac{\xi^2+\lambda\omega^2/\nu}{\sqrt{1-4\lambda\nu\xi^2}}sh\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)\right]d\xi, \tag{35}$$

where A and B are given by Eq. (15).

In a similar fashion, we obtain the shear stress

$$\tau_c(y,t)=\frac{fe^{-By}}{\sqrt{1+\lambda^2\omega^2}}\cos(\omega t-yA-\psi)-\frac{f}{1+\lambda^2\omega^2}e^{-\frac{t}{\lambda}}+\frac{2f}{\pi}\left(\frac{\omega}{\nu}\right)^2e^{-\frac{t}{2\lambda}}\int_0^\infty\frac{\sin(y\xi)}{\xi\left[\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)^2+\left(\frac{\omega}{\nu}\right)^2\right]}\times\left[ch\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)-\frac{1+2\xi^2\left(\frac{\nu}{\omega}\right)^2(\xi^2-\lambda\omega^2/\nu)}{\sqrt{1-4\lambda\nu\xi^2}}\times sh\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)\right]d\xi, \tag{36}$$

corresponding to the cosine oscillation of the shear. The starting shear stresses given by Eqs. (35) and (36), are also presented as a sum of steady-state and transient solutions. Furthermore, the steady-state solutions can be written under the condensed form

$$\tau(y,t)=\frac{fe^{-By}}{\sqrt{1+\lambda^2\omega^2}}\sin(\omega t-yA-\psi+\gamma),\quad \gamma\in\left\{0,\ \frac{\pi}{2}\right\}. \tag{37}$$

Indeed, making $\gamma=0$ or $\gamma=\frac{\pi}{2}$ in Eq. (37), the component parts $\tau_{ss}(y,t)$ and $\tau_{cs}(y,t)$ corresponding to the sine or the cosine oscillations respectively are obtained.

4 Particular Case $\lambda\to 0$ (Newtonian Fluids)

By making $\lambda\to 0$ into Eqs. (15), (22) and (23) and using the limits,

$$\lim_{\lambda\to 0}e^{\frac{-t}{2\lambda}}ch\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)=\lim_{\lambda\to 0}e^{\frac{-t}{2\lambda}}sh\left(\frac{\sqrt{1-4\lambda\nu\xi^2}}{2\lambda}t\right)=\frac{1}{2}e^{-\nu\xi^2t},$$

the known solutions [11]—Eqs. (20) and (22)

$$V_{sN}(y,t)=\frac{f}{\mu}\sqrt{\frac{\nu}{\omega}}e^{-y\sqrt{\frac{\omega}{2\nu}}}\sin\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}+\frac{3\pi}{4}\right)-\frac{2f}{\pi\mu}\frac{\omega}{\nu}\int_0^\infty\frac{\cos(y\xi)}{\xi^4+\left(\frac{\omega}{\nu}\right)^2}e^{-\nu\xi^2t}d\xi, \tag{38}$$

$$V_{cN}(y,t)=\frac{f}{\mu}\sqrt{\frac{\nu}{\omega}}e^{-y\sqrt{\frac{\omega}{2\nu}}}\cos\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}+\frac{3\pi}{4}\right)+\frac{2f}{\pi\mu}\int_0^\infty\frac{\xi^2\cos(y\xi)}{\xi^4+\left(\frac{\omega}{\nu}\right)^2}e^{-\nu\xi^2t}d\xi, \tag{39}$$

corresponding to the Newtonian fluids performing the same motion are recovered. Similarly, from Eqs. (35) and (36) we recover the adequate shear stresses [11, Eqs. (21) and (23)].

$$\tau_{sN}(y,t)=fe^{-y\sqrt{\frac{\omega}{2\nu}}}\sin\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}\right)+\frac{2f\omega}{\pi\nu}\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^4+\left(\frac{\omega}{\nu}\right)^2}e^{-\nu\xi^2t}d\xi, \tag{40}$$

$$\tau_{cN}(y,t)=fe^{-y\sqrt{\frac{\omega}{2\nu}}}\cos\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}\right)-\frac{2f}{\pi}\int_0^\infty\frac{\xi^3\sin(y\xi)}{\xi^4+\left(\frac{\omega}{\nu}\right)^2}e^{-\nu\xi^2t}d\xi. \tag{41}$$

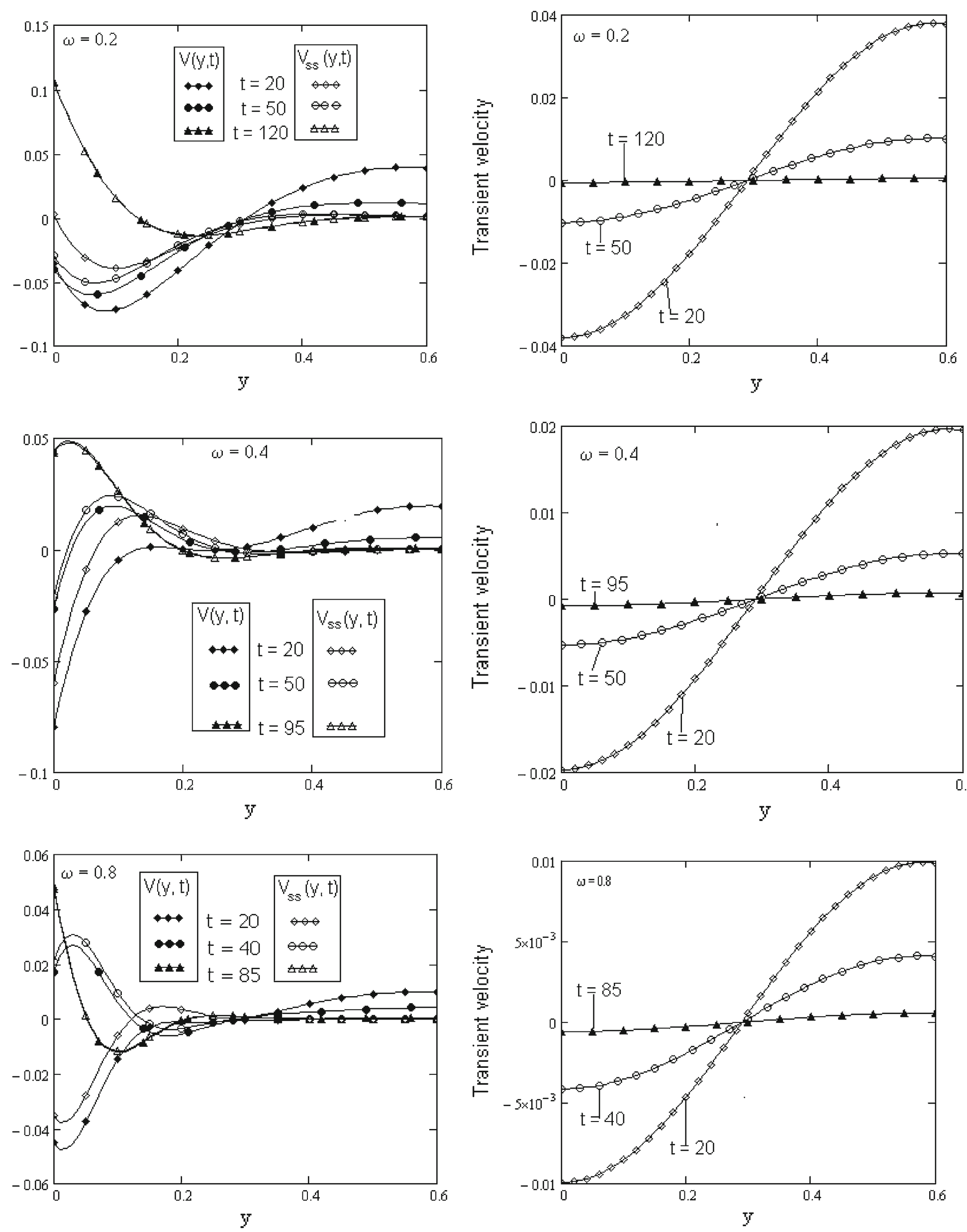

Fig. 1 The required time to reach the steady-state for sine oscillations of the shear stress for $f = 2$, $\upsilon = 0.001457$, $\mu = 1.48$, $\lambda = 0.5$ and different values of t and ω. $V(y,t)$ is the velocity given by Eq. (22) and $V_{ss}(y,t)$ is the steady-state velocity given by Eq. (24)

5 Numerical Results

In this work, the unsteady motion of an incompressible Maxwell fluid due to an infinite plate that applies an oscillating shear to the fluid is studied using integral transforms. The starting solutions that have been obtained, given by Eqs. (22), (23), (35) and (36), satisfy all imposed initial and boundary conditions and can easily be reduced to the similar solutions (38)–(41) for Newtonian fluids performing the same motion. They are presented as a sum between steady-state and transient solutions and describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the motion of the fluid is described by the steady-state solutions that are periodic in time and independent of the initial conditions. However, they satisfy the governing equations and the boundary conditions. Furthermore, as it was to be expected, the steady-state solutions corresponding to sine and cosine oscillations of the shear stress differ by a phase shift.

Generally speaking, the starting solutions for unsteady motions of fluids are important for those who need to eliminate the transients from their rheological measurements. Consequently, an important problem regarding the technical relevance of these solutions is to approximate the time after which the fluid is moving according to the steady-state solutions only. More exactly, in practice, it is necessary to know the required time to reach the steady-state. This time, for sine and cosine oscillations of the shear

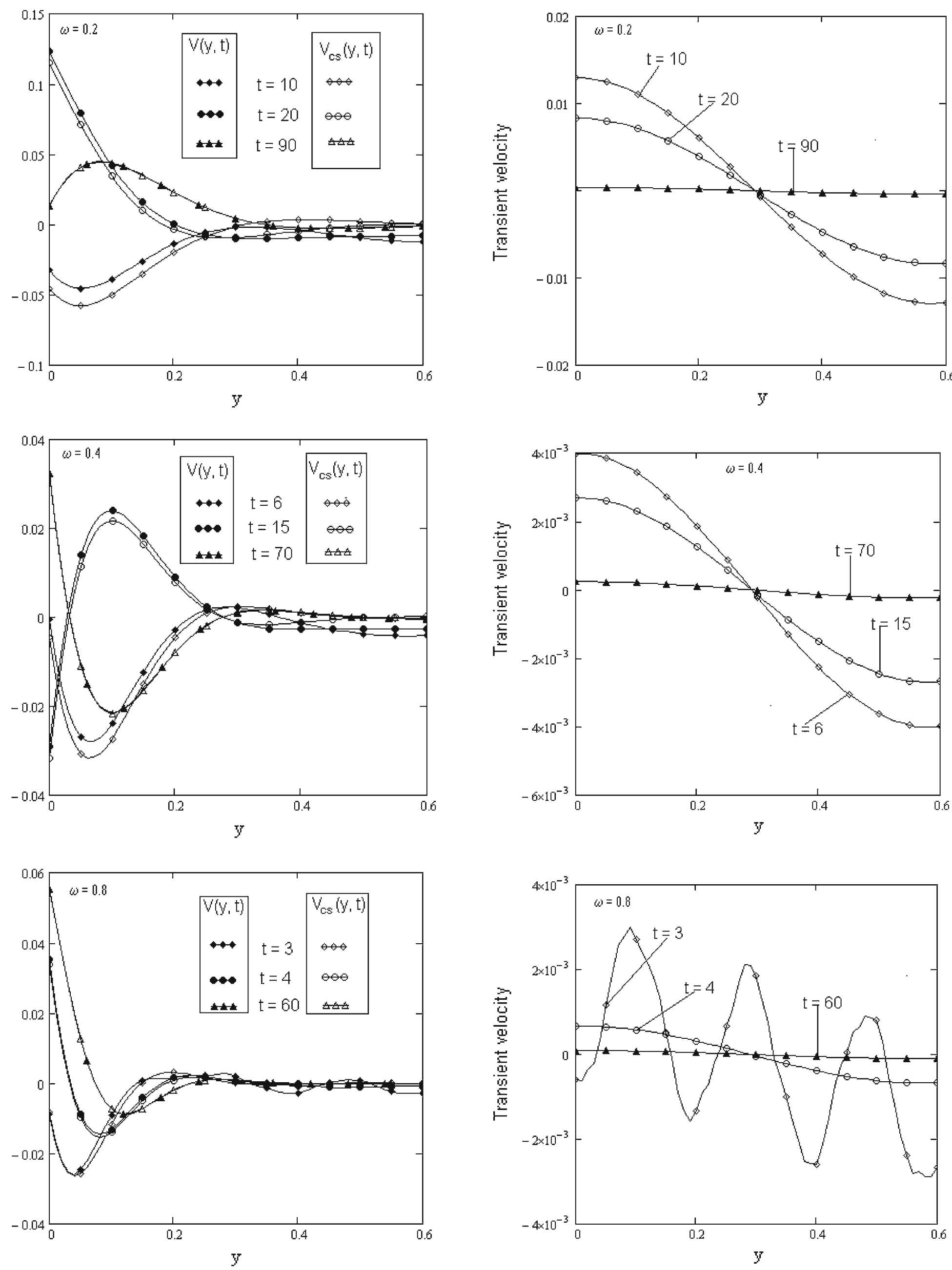

Fig. 2 The required time to reach the steady-state for cosine oscillations of the shear stress for $f = 2$, $\upsilon = 0.001457$, $\mu = 1.48$, $\lambda = 0.5$ and different values of t and ω. $V(y, t)$ is the velocity given by Eq. (23) and $V_{ss}(y, t)$ is the steady-state velocity given by Eq. (25)

stress is determined in Figs. 1, 2, 3, 4 for several values of the frequency ω and the relaxation time λ. For completeness, as well as for a check of the main graphs, the diagrams of the corresponding transient components are also plotted on the second column for the same values of the material constants. Figure 1 contains the diagrams of velocity $V(y, t)$ given by Eq. (22), the permanent solution given $V_{ss}(y, t)$ by Eq. (24) and the transient solution $V_{ts}(y, t) = V(y, t) - V_{ss}(y, t)$ for the sinusoidal oscillations of the shear rate. These diagrams were plotted versus the spatial coordinate y for different values of the frequency ω and time t. Comparing these diagrams, we obtain the values of time t for which the difference between the velocity $V(y, t)$ and the steady-state velocity $V_{ss}(y, t)$ is insignificant (hence, the transient solution can be neglected). For example, if $\omega = 0.4\,\mathrm{s}^{-1}$, the fluid flows according to the permanent solution after 95 s. Also, from these diagrams, we find that the required time to reach the steady-state decreases if ω increases.

In Fig. 2, we have plotted the velocity $V(y, t)$given by Eq. (23), the permanent solution $V_{tc}(y, t)$given by Eq. (25) and the transient solution $V_{tc}(y, t) = V(y, t) - V_{cs}(y, t)$ in the case of co-sinusoidal oscillations of the shear rate. These diagrams were plotted versus y for different

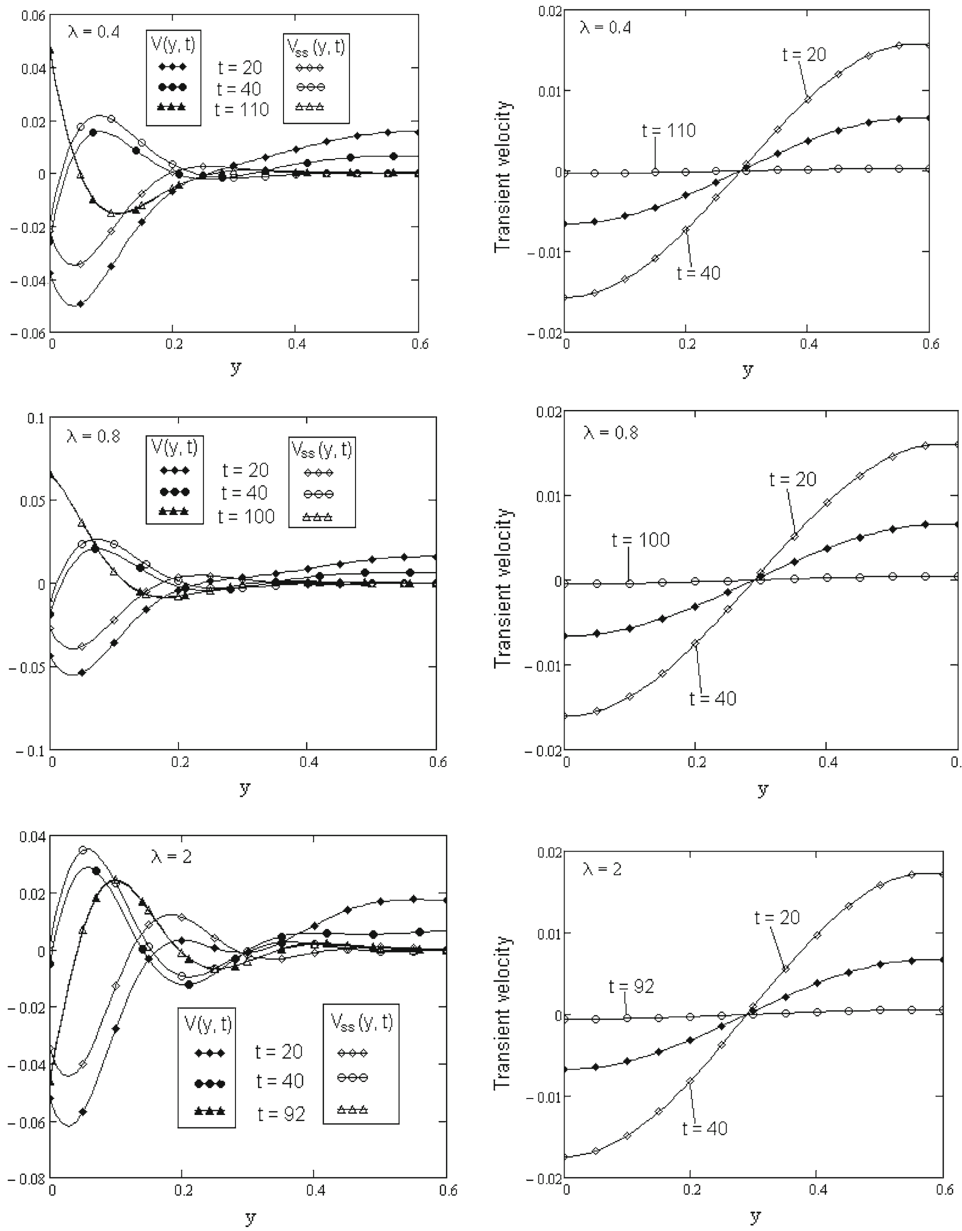

Fig. 3 The required time to reach the steady-state for sine oscillations of the shear stress for $f = 2$, $\upsilon = 0.001457$, $\mu = 1.48$, $\omega = 0.5$ and different values of t and λ. $V(y, t)$ is the velocity given by Eq. (22) and $V_{ss}(y, t)$ is the steady-state velocity given by Eq. (24)

values of the frequency ω and time t. It is easy to find the time for which the velocity $V(y, t)$ can be approximated by the permanent solution (the transient solution can be neglected). Similar to the case of sinusoidal oscillations, the required time to reach the steady-state is decreasing for increasing ω. Figure 3 contains the diagrams of velocity, permanent velocity and transient velocity for sine oscillations of the shear, versus y and for different values of the time t and the relaxation time λ. From these diagrams, we find the required time to reach the steady-state. This time decreases if λ increases. Figure 4 contains similar results as in Fig. 3 but for cosine oscillations of the shear.

6 Conclusions

The motion of a Maxwell fluid over an infinite plate that applies an oscillating shear to the fluid is studied by means of integral transforms. Closed-forms of solutions are written as a sum between steady-state (permanent solutions) and transient solutions. For large values of time t transient solutions tend to zero and the fluid flows according to the permanent solutions. The required time to reach steady-state was determined using numerical results and graphical illustrations. The main outcomes are the given below.

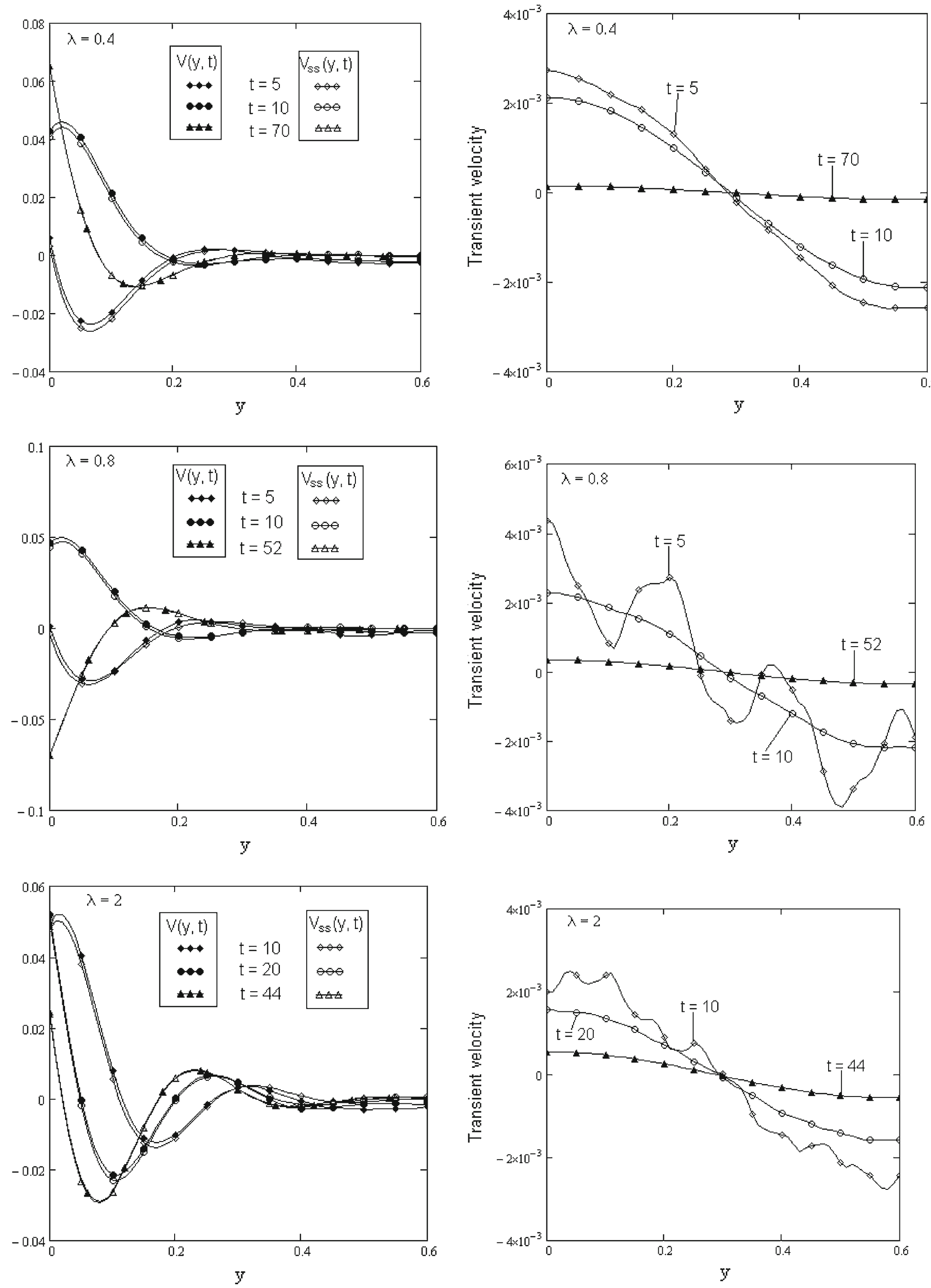

Fig. 4 The required time to reach the steady-state for cosine oscillations of the shear stress for $f = 2$, $\upsilon = 0.001457$, $\mu = 1.48$, $\omega = 0.5$ and different values of t and λ. $V(y,t)$ is the velocity given by Eq. (23) and $V_{ss}(y,t)$ is the steady-state velocity given by Eq. (25)

The relaxation time as well as the frequency have significant influence on the motion. This influence seems to be stronger for the motion due to the sinusoidal oscillations of the shear.

The required time to reach the steady-state is lower for co-sinusoidal oscillations in comparison with the sine oscillations of the shear stress.

The required time to get steady-state decreases if frequency or the relaxation time increases.

The required time to reach the steady-state is lower for Maxwell fluids in comparison with Maxwell fluids.

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Appendix

$$\int_0^\infty \frac{\cos(mx)}{\left(x^2-b^2\right)^2+c^2}\mathrm{d}x = \frac{\pi \mathrm{e}^{-mB}}{2c\left(A^2+B^2\right)}\left[A\cos(mA)+B\sin(mA)\right] \quad \text{(A.1)}$$

$$\int_0^\infty \frac{\left(x^2-b^2\right)\cos(mx)}{\left(x^2-b^2\right)^2+c^2}\mathrm{d}x = \frac{\pi \mathrm{e}^{-mB}}{2\left(A^2+B^2\right)}\left[B\cos(mA)-A\sin(mA)\right], \quad \text{(A.2)}$$

$$\int_0^\infty \frac{\sin(y\xi)}{\xi} = \pi/2, \quad \text{(A.3)}$$

$$\int_0^\infty \frac{\left(x^2-b^2\right))\sin(mx)}{x\left[\left(x^2-b^2\right)^2+c^2\right]}\mathrm{d}x = \frac{\pi}{2\left(b^4+c^2\right)}\left\{-b^2+\left[b^2\cos(mA)+c\sin(mA)\right].exp\left(-mB\right)\right\}, \quad \text{(A.4)}$$

$$\int_0^\infty \frac{\sin(mx)}{x\left[\left(x^2-b^2\right)^2+c^2\right]}\mathrm{d}x = \frac{\pi}{2c\left(b^4+c^2\right)}\left\{c+\left[b^2\sin(mA)-c\cos(mA)\right].exp(-mB)\right\}, \quad \text{(A.5)}$$

where $b=\omega\sqrt{\frac{\lambda}{\nu}}$, $c=\frac{\omega}{\nu}$, $A=\sqrt{\frac{\sqrt{b^4+c^2}+b^2}{2}}$ and $B=\sqrt{\frac{\sqrt{b^4+c^2}-b^2}{2}}$.

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