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EXACT SOLUTIONS FOR OSCILLATING MOTION OF A SECOND-GRADE FLUID ALONG AN EDGE WITH MIXED BOUNDARY CONDITIONS

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Exact Solutions for Oscillating Motion of a Second-Grade Fluid along an Edge with Mixed Boundary Conditions

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Exact solutions corresponding to the oscillating motion of a second-grade fluid along the inside of an edge are established by means of integral transforms. The motion of the fluid is due to the walls of the edge. One of them applies an oscillating shear to the fluid and the other one is subject to an oscillatory motion in its plane. The solutions that have been obtained are presented in the form of simple and multiple integrals and satisfy all imposed initial and boundary conditions. Finally, some characteristics of the fluid motion, as well as the influence of Reynolds number on the velocity, are graphically underlined.

Keywords Exact solutions; Second-grade fluid; Shear stresses; Unsteady motion; Velocity field

Introduction

The motion of a fluid along the inside of an edge has attracted much attention due to its practical importance and fundamental value for theory. An elegant solution corresponding to the Rayliegh–Stokes problem for an edge, was given by Zierep (1979) for Newtonian fluids. This solution has been extended to non-Newtonian fluids by Fetecau (2002) and for the fluids with fractional derivatives by Khan (2009) and Corina Fetecau et al. (2009). Interesting results have been also obtained by Fetecau and Corina Fetecau (2004) and Fetecau and Prasad (2005) for the flow induced by a constantly accelerating edge in non-Newtonian fluids and by Nadeem (2007) for periodic flows of fractional Oldroyd-B fluids through an edge. However, there is no result in the literature in which the shear stress is given on the edge or one of its sides. The first exact solutions for the motions of second-grade fluids in which the shear stress is given on a part of the boundary seem to be those of Bandelli and Rajagopal (1995). They have been recently extended to more general fluids (Jamil et al., 2011) and other interesting results on hydromagnetic bounary layer flow over a moving surface have been obtained by Subhas Abel et al. (2010) and Pantokratoras (2011).

Our aim here is to study the oscillating motion of a second-grade fluid along the inside of an edge whose settlement leads to a mixed initial boundary-value problem.

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More exactly, a side of the edge applies an oscillating shear to the fluid and the other one is oscillating in its plane. An interesting aspect of the problem being studied is that unlike the usual no-slip boundary condition that is used, a secondary condition on the shear stress is used. This is very important since in some problems what is specified is the force applied on the boundary. It is also important to bear in mind that the no-slip boundary condition may not be necessarily applicable for flows of polymeric fluids that can slip or slide on the boundary. Thus, a shear stress boundary condition can be particularly meaningful. Moreover, in addition to being a study of a time-dependent problem, it leads to exact solutions. Such exact solutions are not common in the literature and in addition to providing an elegant resolution of the problem, they are an important check for numerical methods that are used to study the flows of such fluids in a complex domain. It is hoped that the current analysis will add to the useful literature concerning second-grade fluids. Finally, some characteristics of the fluid motion are underlined by graphical illustrations.

Governing Equations

The constitutive equation of an incompressible second-grade fluid is (Dunn and Rajagopal, 1995)

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ is the extra-stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors, μ is the dynamic viscosity, and α_1, α_2 are the normal stress moduli. The Clausius-Duhen inequality and the assumption that the Helmhotz free energy is minimum in equilibrium provide the following restrictions (Dunn and Fosdick, 1974)

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0$$

A comprehensive discussion on these restrictions can be found in the work of Dunn and Rajagopal (1995). The sign of the material moduli α_1 and α_2 was the subject of much controversy. In experiments on several non-Newtonian fluids, the last two restrictions have not been confirmed. However, the conclusion was that the fluids that have been tested are not fluids of second grade and they are characterized by a different constitutive structure. If the second inequality is reversed, so that $\alpha_1 < 0$, then the fluid model considered leads to an unacceptable general instability (Dunn and Fosdick, 1974). The flows to be here considered have the velocity field (Fetecau and Fetecau, 2004; Fetecau and Prasad, 2005)

$$\mathbf{V} = \mathbf{V}(y, z, t) = u(y, z, t)\mathbf{i} \tag{2}$$

where $\mathbf{i}$ denotes the unit vector along the x-direction of the Cartesian coordinates system x, y, and z. For these flows, the constraint of incompressibility is automatically satisfied. Using Equations (1) and (2) and the balance of linear momentum, in the absence of body forces and a pressure gradient in the x-direction, we obtain the following partial differential equations (Vieru et al., 2011):

$$\tau_1\left(y, z, t\right) = (\mu + \alpha_1 \frac{\partial}{\partial t}\Big) \frac{\partial u(y, z, t)}{\partial y}, \quad \tau_2(y, z, t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \frac{\partial u(y, z, t)}{\partial z} \tag{3}$$

$$\frac{\partial u(y,z,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] u(y,z,t) \tag{4}$$

where $\tau_1(y, z, t) = S_{xy}(y, z, t)$ and $\tau_2(y, z, t) = S_{xz}(y, z, t)$ are the nontrivial shear stresses, $\alpha = \alpha_1/\rho$, ρ is the constant density of the fluid, and $\nu = \mu/\rho$ is the kinematic viscosity. In the following the governing Equation (4) together with suitable initial and boundary conditions will be solved by means of the Laplace and the Fourier transforms.

Statement of the Problem and Solutions

Let us consider an incompressible second-grade fluid at rest occupying the space of the first dial of a rectangular edge (Fetecau and Fetecau, 2004; Fetecau and Prasad, 2005) ($-\infty < x < \infty$, $y \geq 0$, $z \geq 0$). After time $t = 0$ a side of the boundary applies an oscillating shear $f_1 \sin(\omega t)$ or $f_1 \cos(\omega t)$ to the fluid while the other part is being subjected to an oscillatory motion in its plane. Owing to the shear the fluid is gradually moved. Its velocity is of the form (2) and the governing equation is given by Equation (4). The appropriate initial and boundary conditions are

$$u(y,z,0) = 0, \quad y, z \geq 0 \tag{5}$$

$$\tau_1(0,z,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \frac{\partial u(y,z,t)}{\partial y}\bigg|_{y=0} = f_1 \sin(\omega_1 t), \quad \text{or} \quad f_1 \cos(\omega_1 t) \quad z > 0,\ t > 0 \tag{6}$$

$$u(y,0,t) = f_2 \sin(\omega_2 t) \quad \text{or} \quad f_2 \cos(\omega_2 t), \quad y > 0,\ t > 0 \tag{7}$$

Furthermore, the natural condition (Fetecau and Fetecau, 2004)

$$u(y,z,t) \to 0, \quad \text{for } y \to \infty \text{ and } z \to \infty \tag{8}$$

has to be also satisfied.

Introducing the dimensionless variables

$$t^\star = \frac{t}{\left(\frac{\alpha}{\nu}\right)}, \quad y^\star = \frac{y}{\left(\frac{\mu f_2}{f_1}\right)}, \quad z^\star = \frac{z}{\left(\frac{\mu f_2}{f_1}\right)}, \quad u^\star = \frac{u}{f_2},$$
$$\tau_1^\star = \frac{\tau_1}{f_1}, \quad \tau_2^\star = \frac{\tau_2}{f_1}, \quad \omega_1^\star = \frac{\omega_1 \alpha}{\nu}, \quad \omega_2^\star = \frac{\omega_2 \alpha}{\nu}$$

into Equations (3) and (4) and neglecting the asterisks, we find that

$$\tau_1(y,z,t) = \left(1 + \frac{\partial}{\partial t}\right)\frac{\partial u(y,z,t)}{\partial y} \quad \tau_2(y,z,t) = v\left(1 + \frac{\partial}{\partial t}\right)\frac{\partial u(y,z,t)}{\partial z} \tag{9}$$

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\left(1 + \frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] u(y,z,t) \tag{10}$$

where $Re = (\frac{\mu f_2}{f_1})^2/\alpha$ is the Reynolds number. The initial and boundary conditions become

$$u(y,z,0) = 0, \quad y, z \geq 0, \tag{11}$$

$$\tau_1(0,z,t) = \left(1 + \frac{\partial}{\partial t}\right) \frac{\partial u(y,z,t)}{\partial y}\Big|_{y=0} = \sin(\omega_1 t), \quad z > 0,\ t > 0 \tag{12}$$

$$u(y,0,t) = \sin(\omega_2 t), \quad y > 0,\ t > 0 \tag{13}$$

$$u(y,z,t) \to 0, \quad \text{for } y \to \infty \text{ and } z \to \infty \tag{14}$$

Calculation of the Velocity Field

In order to avoid repetition we present all calculations, as well as the final results, for the case of sine oscillations. Applying the Laplace transform to Equations (10) and (12)–(14) and using Equation (11), we get (Sneddon, 1955; Debnath and Bhatta, 2007)

$$q\overline{u}(y,z,q) = \frac{1+q}{Re}\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\overline{u}(y,z,q) \tag{15}$$

$$(1+q)\frac{\partial \overline{u}(y,z,q)}{\partial y}\bigg|_{y=0} = \frac{\omega_1}{q^2+\omega_1^2}, \quad \overline{u}(y,0,q) = \frac{\omega_2}{q^2+\omega_2^2} \tag{16}$$

$$\overline{u}(y,z,q) \to 0, \quad \text{for } y \to \infty \text{ and } z \to \infty \tag{17}$$

where $\overline{u}(y,z,q)$ is the Laplace transform of the function $u(y,\ z,\ t)$.

Multiplying Equation (15) by $\sqrt{\frac{2}{\pi}}\sin(z\eta)$ and integrating with respect to z from 0 to ∞, we obtain

$$\frac{\partial^2 \overline{u}_s(y,\eta,q)}{\partial y^2} - \frac{(\eta^2 + Re)q + \eta^2}{q+1}\overline{u}_s(y,\eta,q) = -\eta\sqrt{\frac{2}{\pi}}\frac{\omega_2}{q^2+\omega_2^2} \tag{18}$$

where $\overline{u}_s(y,\eta,q)$ is the Fourier sine transform of the function $\overline{u}(y,z,q)$.

The general solution of Equation (18) is

$$\overline{u}_s(y,\eta,q) = C_1 e^{-y\sqrt{W(\eta,q)}} + C_2 e^{y\sqrt{W(\eta,q)}} + \eta\sqrt{\frac{2}{\pi}}\frac{1}{W(\eta,q)}\frac{\omega_2}{q^2+\omega_2^2} \tag{19}$$

where C_1, C_2 are arbitrary constants and

$$W(\eta,q) = \frac{(\eta^2+Re)q+\eta^2}{q+1}$$

The boundary conditions $(16)_1$ and (17) also imply

$$\frac{\partial \overline{u}_s(0,\eta,q)}{\partial y} = \sqrt{\frac{2}{\pi}}\frac{\omega_1}{q^2+\omega_1^2}\frac{1}{q+1}\frac{1}{\eta}, \overline{u}_s(y,\eta,q) \to 0 \quad \text{as } y \to \infty \tag{20}$$

Combining Equations (19) and (20), we find that

$$\overline{u}_s(y,\eta,q) = \eta\sqrt{\frac{2}{\pi}}\frac{1}{W(\eta,q)}\frac{\omega_2}{q^2+\omega_2^2} - \frac{1}{\eta}\sqrt{\frac{2}{\pi}}\frac{\omega_1}{q^2+\omega_1^2}\frac{1}{q+1}\frac{e^{-y\sqrt{W(\eta,q)}}}{\sqrt{W(\eta,q)}} \tag{21}$$

Denoting by

$$\overline{u}_{s1}(\eta,q) = \eta\sqrt{\frac{2}{\pi}}\frac{q+1}{(\eta^2+Re)q+\eta^2}\frac{\omega_2}{q^2+\omega^2}$$

$$\overline{u}_{s2}(\eta,q) = \frac{1}{\eta}\sqrt{\frac{2}{\pi}}\frac{\omega_1}{q^2+\omega_1^2}\frac{1}{q+1}\sqrt{W(\eta,q)}, \quad \overline{u}_{s3}(y,\eta,q) = \frac{e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}$$

it results that

$$u_s(y,\eta,t) = u_{s1}(\eta,t) - u_{s2}(\eta,t) * u_{s3}(y,\eta,t) \tag{22}$$

where $*$ denotes the convolution product and $u_{s1}(\eta, t)$, $u_{s2}(\eta, t)$, and $u_{s3}(y, \eta, t)$ are the inverse Laplace transforms of $\overline{u}_{s1}(\eta,q)$, $\overline{u}_{s2}(\eta,q)$, and $\overline{u}_{s3}(y,\eta,q)$, respectively. In order to determine $u_{s1}(\eta, t)$ and $u_{s2}(\eta, t)$, we write $\overline{u}_{s1}(\eta,q)$ and $\overline{u}_{s2}(\eta,q)$ under suitable forms:

$$\overline{u}_{s1}(\eta,q) = \sqrt{\frac{2}{\pi}}\frac{\omega_2}{q^2+\omega_2^2}\frac{\eta}{(\eta^2+Re)}\left[1+\frac{Re}{(\eta^2+Re)}\frac{1}{q+\frac{\eta^2}{(\eta^2+Re)}}\right] \tag{23}$$

$$\begin{aligned}\overline{u}_{s2}(\eta,q) = {} & \sqrt{\frac{2}{\pi}}\frac{\omega_1}{q^2+\omega_1^2}\left[(\eta^2+Re) - \frac{Re}{q+1}\right] \\ & \times \frac{1}{\eta\sqrt{\eta^2+Re}}\frac{1}{\sqrt{(q+\frac{2\eta^2+Re}{2(\eta^2+Re)})^2 - (\frac{Re}{2(\eta^2+Re)})^2}}\end{aligned} \tag{24}$$

Applying the inverse Laplace transform to the functions $\overline{u}_{s1}$, $\overline{u}_{s2}$, $\overline{u}_{s3}$ and using Equations (A1)–(A6) from the Appendix, we find (Roberts and Kaufman, 1968)

$$\begin{aligned}u_{s1}(\eta,t) = {} & \sqrt{\frac{2}{\pi}}\frac{\eta}{(\eta^2+Re)}\sin(\omega_2 t) \\ & + \sqrt{\frac{2}{\pi}}\frac{Re\eta}{(\eta^2+Re)^2}\int_0^t \sin(\omega_2 s)e^{-\frac{\eta^2}{\eta^2+Re}(t-s)}ds\end{aligned} \tag{25}$$

$$u_{s2}(\eta,t)=\sqrt{\frac{2}{\pi}}\frac{\sqrt{\eta^2+Re}}{\eta}\int_0^t \sin(\omega_1 s)I_o\left[\frac{Re(t-s)}{2(\eta^2+Re)}\right] \times e^{-\frac{2\eta^2+Re}{2(\eta^2+Re)}(t-s)}ds-\sqrt{\frac{2}{\pi}}\frac{Re}{\eta\sqrt{\eta^2+Re}}\int_0^t\int_0^\sigma \sin(\omega_1 s) \times I_o\left[\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right]e^{-\frac{2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-(t-\sigma)}ds d\sigma \tag{26}$$

$$u_{s3}(\eta,t)=\int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)I_1\left[2\sqrt{utRe}\right]\sqrt{\frac{uRe}{t}}e^{-u(\eta^2+Re)-t}du+\delta(t)\frac{e^{-y\sqrt{\eta^2+Re}}}{\eta^2+Re} \tag{27}$$

where $\delta(.)$ is the Dirac delta function.

Introducing Equations (25)–(27) in Equation (22), it results that

$$\begin{aligned}
u_s(y,\eta,t)=&\sqrt{\frac{2}{\pi}}\frac{\eta}{\eta^2+Re}\sin(\omega_2 t)+\sqrt{\frac{2}{\pi}}\frac{\eta Re}{(\eta^2+Re)^2}\\
&\times\int_0^t \exp\left(-\frac{\eta^2}{\eta^2+Re}(t-s)\right)\sin(\omega_2 s)ds-\sqrt{\frac{2}{\pi}}\frac{e^{-y\sqrt{\eta^2+Re}}}{\eta\sqrt{\eta^2+Re}}\\
&\times\int_0^t I_o\left(\frac{Re(t-s)}{2(\eta^2+Re)}\right)\exp\left[\left(-\frac{2\eta^2+Re}{2(\eta^2+Re)}(t-s)\right)\right]\sin(\omega_1 s)ds\\
&+\sqrt{\frac{2}{\pi}}Re\frac{e^{-y\sqrt{\eta^2+Re}}}{\eta(\eta^2+Re)^{3/2}}\int_0^t\int_0^\sigma I_o\left(\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right)\\
&\times\exp\left[\left(-\frac{2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-(t-\sigma)\right)\right]\sin(\omega_1 s)ds\,d\sigma\\
&-\sqrt{\frac{2}{\pi}}\frac{\sqrt{\eta^2+Re}}{\eta}\int_0^\infty\int_0^t\int_0^\sigma erfc\left(\frac{y}{2\sqrt{u}}\right)I_o\left(\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right)\\
&\times I_1\left[2\sqrt{uRe(t-\sigma)}\right]\sqrt{\frac{uRe}{t-\sigma}}\sin(\omega_1 s)\\
&\times\exp\left[\left(-\frac{2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-(t-\sigma)-u(\eta^2+Re)\right)\right]ds\,d\sigma\,du\\
&+\sqrt{\frac{2}{\pi}}\frac{Re}{\eta\sqrt{\eta^2+Re}}\int_0^\infty\int_0^t\int_0^\sigma\int_0^\tau erfc\left(\frac{y}{2\sqrt{u}}\right)\\
&\times I_o\left(\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right)I_1\left[2\sqrt{uRe(t-\tau)}\right]\\
&\times\exp\left[\left(-\frac{2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-(t-\sigma)-u(\eta^2+Re)\right)\right]\\
&\times\sqrt{\frac{uRe}{t-\tau}}\sin(\omega_1 s)ds\,d\sigma\,d\tau\,du
\end{aligned} \tag{28}$$

Applying the inverse Fourier sine transform to Equation (28) and using Equations (A7)–(A9) from the Appendix, we obtain the velocity field

$$
\begin{aligned}
u_s(y,z,t) &= e^{-Bz}\sin(\omega_2 t - Az) + e^{-Bz}\sin(Az)e^{-\frac{\eta^2}{\eta^2+Re}t} \\
&-\frac{2}{\pi}\int_0^\infty \int_0^t \frac{e^{-y\sqrt{\eta^2+Re}}}{\eta(\sqrt{\eta^2+Re})}\sin(\eta z) I_o\left(\frac{Re(t-s)}{2(\eta^2+Re)}\right) \\
&\times \exp\left[\left(-\frac{2\eta^2+Re}{2(\eta^2+Re)}(t-s)\right)\right]\sin(\omega_1 s)ds\,d\eta \\
&+\frac{2Re}{\pi}\int_0^\infty \int_0^t \int_0^\sigma \frac{e^{-y\sqrt{\eta^2+Re}}}{\eta(\eta^2+Re)^{3/2}}\sin(\eta z) I_o\left(\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right) \\
&\times \exp\left[-\frac{2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-(t-\sigma)\right]\sin(\omega_1 s)ds\,d\sigma\,d\eta \\
&-\frac{2}{\pi}\int_0^\infty \int_0^\infty \int_0^t \int_0^\sigma \frac{\sqrt{\eta^2+Re}}{\eta}\sin(\eta z) erfc(\frac{y}{2\sqrt{u}}) I_o\left(\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right) \\
&\times I_1\left[2\sqrt{uRe(t-\sigma)}\right]\exp\left[-\frac{2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-(t-\sigma)-u(\eta^2+Re)\right] \\
&\times \sqrt{\frac{uRe}{t-\sigma}}\sin(\omega_1 s)ds\,d\sigma\,du\,d\eta \\
&+\frac{2Re}{\pi}\int_0^\infty \int_0^\infty \int_0^t \int_0^\sigma \int_0^\tau \frac{\sin(\eta z)}{\eta\sqrt{(\eta^2+Re)}} erfc\left(\frac{y}{2\sqrt{u}}\right) \\
&\times I_o\left(\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right) I_1\left[2\sqrt{uRe(t-\tau)}\right]\times \\
&\times \exp\left[\left(-\frac{2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-(t-\sigma)-u(\eta^2+Re)\right)\right] \\
&\times \sqrt{\frac{uRe}{t-\tau}}\sin(\omega_1 s)ds\,d\sigma\,d\tau\,du\,d\eta
\end{aligned}
\tag{29}
$$

where A and B are defined in the Appendix.

Calculation of the Shear Stresses

Applying the Laplace transform to Equation $(9)_1$ and the Fourier sine transform with respect to z to the obtained result, we find that

$$\overline{\tau}_{s1}(y,\eta,q) = (1+q)\frac{\partial \overline{u}_s(y,\eta,q)}{\partial y} \tag{30}$$

Introducing $\overline{u}_s(y,\eta,q)$ from Equation (21) into this relation, we get

$$\overline{\tau}_{s1}(y,\eta,q) = \sqrt{\frac{2}{\pi}}\frac{\omega_1}{q^2+\omega_1^2}\frac{1}{\eta}e^{-y\sqrt{W(\eta,q)}} \tag{31}$$

which can be written under the form

$$\overline{\tau}_{s1}(y,\eta,q) = \overline{T}_{s1}(\eta,q) - \overline{T}_{s2}(\eta,q)\overline{T}_{s3}(y,\eta,q), \tag{32}$$

where

$$\overline{T}_{s1}(\eta, q) = \sqrt{\frac{2}{\pi}}\frac{\omega_1}{q^2+\omega_1^2}\frac{1}{\eta},$$
$$\overline{T}_{s2}(\eta, q) = \sqrt{\frac{2}{\pi}}\frac{\eta^2+Re}{\eta}\frac{\omega_1}{q^2+\omega_1} - \sqrt{\frac{2}{\pi}}\frac{Re}{\eta}\frac{\omega_1}{q^2+\omega_1}\frac{1}{q+1} \tag{33}$$

and

$$\overline{T}_{s3}(y, \eta, q) = \frac{1-e^{-y\sqrt{W(\eta,q)}}}{W(\eta, q)} \tag{34}$$

Applying the inverse Laplace transform to Equations (33) and (34) and using Equations (A1)–(A7) from Appendix, we obtain

$$T_{s1}(\eta, t) = \sqrt{\frac{2}{\pi}}\frac{1}{\eta}\sin(\omega_1 t) \tag{35}$$

$$T_{s2}(\eta, t) = \sqrt{\frac{2}{\pi}}\frac{\eta^2+Re}{\eta}\sin(\omega_1 t) - \sqrt{\frac{2}{\pi}}\frac{Re}{\eta}\int_0^t \sin(\omega_1 s)e^{-(t-s)}ds \tag{36}$$

and

$$T_{s3}(y, \eta, t) = \int_0^\infty erf(\frac{y}{2\sqrt{u}})I_1\left[2\sqrt{utRe}\right]\sqrt{\frac{uRe}{t}}$$
$$\times \exp\left[-u(\eta^2+Re)-t\right]du + \delta(t)\frac{1-e^{-y\sqrt{\eta^2+Re}}}{\eta^2+Re} \tag{37}$$

Now applying the inverse Laplace transform to Equation (32) and using Equations (35)–(37) and the convolution theorem for the last term, we find that

$$\tau_{s1}(y, \eta, t) = \sqrt{\frac{2}{\pi}}\sin(\omega_1 t)\frac{e^{-y\sqrt{\eta^2+Re}}}{\eta} + Re\sqrt{\frac{2}{\pi}}.\frac{1-e^{-y\sqrt{\eta^2+Re}}}{\eta(\eta^2+Re)}\int_0^t \sin(\omega_1 s)e^{-(t-s)}ds$$
$$-\sqrt{\frac{2}{\pi}}\frac{\eta^2+Re}{\eta}\int_0^\infty\int_0^t \sin(\omega_1 s)erf\left(\frac{y}{2\sqrt{u}}\right)I_1\left[2\sqrt{Reu(t-s)}\right]$$
$$\times\sqrt{\frac{uRe}{t-s}}\exp\left[-u(\eta^2+Re)-(t-s)\right]ds\,du$$
$$+\sqrt{\frac{2}{\pi}}\frac{Re}{\eta}\int_0^\infty\int_0^t\int_0^\sigma \sin(\omega_1 s)erf\left(\frac{y}{2\sqrt{u}}\right)I_1\left[2\sqrt{uRe(t-\sigma)}\right]$$
$$\times\sqrt{\frac{uRe}{t-\sigma}}\exp\left[-u(\eta^2+Re)-(t-s)\right]ds\,d\sigma\,du \tag{38}$$

Finally, applying the inverse Fourier sine transform to Equation (38) we obtain the shear stress $\tau_1(y, z, t)$ under the form

$$\begin{aligned}\tau_{s1}(y,z,t) &= \frac{2}{\pi}\sin(\omega_1 t)\int_0^\infty \frac{e^{-y\sqrt{\eta^2+Re}}\sin(\eta z)}{\eta}d\eta + \frac{2Re}{\pi}\int_0^\infty\int_0^t \frac{1-e^{-y\sqrt{\eta^2+Re}}}{\eta(\eta^2+Re)} \\ &\times \sin(\eta z)\sin(\omega_1 s)e^{-(t-s)}ds\,d\eta - \frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t \frac{\eta^2+Re}{\eta} \\ &\times \sin(\eta z)\sin(\omega_1 s)erf\left(\frac{y}{2\sqrt{u}}\right)I_1\left[2\sqrt{uRe(t-s)}\right]\sqrt{\frac{uRe}{t-s}} \\ &\times \exp\left[-u(\eta^2+Re)-(t-s)\right]ds\,du\,d\eta \\ &+ \frac{2Re}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma \frac{\sin\eta z}{\eta}\sin(\omega_1 s)erf\left(\frac{y}{2\sqrt{u}}\right)I_1\left[2\sqrt{uRe(t-\sigma)}\right] \\ &\times \sqrt{\frac{uRe}{t-\sigma}}\exp\left[-u(\eta^2+Re)-(t-s)\right]ds\,d\sigma\,du\,d\eta\end{aligned} \tag{39}$$

In order to determine the second shear stress, from Equation $(9)_2$ we have

$$\overline{\tau}_{s2}(y,z,q) = (1+q)\frac{\partial\overline{u}(y,z,q)}{\partial z} \tag{40}$$

Applying the inverse Fourier sine transform to Equation (21) and introducing the result in Equation (40) it results that

$$\begin{aligned}\overline{\tau}_{s2}(y,z,q) &= \frac{2}{\pi}\int_0^\infty \eta^2\cos(\eta z)\frac{\omega_2}{q^2+\omega_2^2}\frac{q+1}{W(\eta,q)}d\eta \\ &- \frac{2}{\pi}\int_0^\infty \cos(\eta z)\frac{\omega_1}{q^2+\omega_1^2}\frac{W(\eta,q)}{\sqrt{W(\eta,q)}}\frac{e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}d\eta\end{aligned} \tag{41}$$

To determine the inverse Laplace transform of the function $\overline{\tau}_{s2}(y,z,q)$ we use the relation

$$\begin{aligned}\overline{\tau}_{s2}(y,z,q) &= \frac{2}{\pi}\int_0^\infty \eta^2\cos(\eta z)\overline{T^\star}_{s1}(\eta,q)d\eta \\ &- \frac{2}{\pi}\int_0^\infty \cos(\eta z)\overline{T^\star}_{s2}(\eta,q)\overline{T^\star}_{s3}(y,\eta,q)d\eta\end{aligned} \tag{42}$$

where

$$\begin{aligned}\overline{T^\star}_{s1}(\eta,q) &= \frac{\omega_2}{q^2+\omega_2^2}\frac{q+1}{W(\eta,q)} \\ &= \frac{\omega_2}{q^2+\omega_2^2}\left[\frac{q}{\eta^2+Re} + \frac{\eta^2+2Re}{(\eta^2+Re)^2} + \frac{(Re)^2}{(\eta^2+Re)^3[q+\frac{\eta^2}{\eta^2+Re}]}\right]\end{aligned} \tag{43}$$

$$\overline{T^{\star}}_{s2}(\eta,q)=\frac{\omega_1}{q^2+\omega_1^2}\frac{W(\eta,q)}{\sqrt{W(\eta,q)}}=\left[\omega_1\sqrt{\eta^2+Re}\frac{q}{q^2+\omega_1^2}+\frac{\eta^2}{\sqrt{\eta^2+Re}}\frac{\omega_1}{q^2+\omega_1^2}\right]\times\frac{1}{\sqrt{(q+\frac{2\eta^2+Re}{2(\eta^2+Re)})^2-(\frac{Re}{2(\eta^2+Re)})^2}} \tag{44}$$

and

$$\overline{T^{\star}}_{s3}(y,\eta,q)=\frac{e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)} \tag{45}$$

Using Equations (A1)–(A7) from Appendix, we can write

$$T^{\star}_{s1}(\eta,t)=\frac{\omega_2}{\eta^2+Re}\cos(\omega_2 t)+\frac{\eta^2+2Re}{(\eta^2+Re)^2}\sin(\omega_2 t)+\frac{Re^2}{(\eta^2+Re)^3}\int_0^t e^{-\frac{\eta^2}{\eta^2+Re}(t-s)}\sin(\omega_2 s)ds \tag{46}$$

$$T^{\star}_{s2}(\eta,t)=\int_0^t\left[\omega_1\sqrt{\eta^2+Re}\cos(\omega_1 s)+\frac{\eta^2}{\sqrt{\eta^2+Re}}\sin(\omega_1 s)\right]\times I_o\left[\frac{Re(t-s)}{2(\eta^2+Re)}\right]e^{-\frac{2\eta^2+Re}{2(\eta^2+Re)}(t-s)}ds \tag{47}$$

$$T^{\star}_{s3}(y,\eta,t)=\int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)I_1\left[2\sqrt{utRe}\right]\sqrt{\frac{uRe}{t}}\times\exp\left[-u(\eta^2+Re)-t\right]du+\delta(t)\frac{e^{-y\sqrt{\eta^2+Re}}}{\eta^2+Re} \tag{48}$$

Applying the inverse Laplace transform to Equation (42), using Equations (46)–(48) and the convolution theorem, we obtain

$$\tau_{s2}(y,z,t)=-\omega_2\sqrt{Re}\cos(\omega_2 t)e^{-z\sqrt{Re}}-\sqrt{Re}\sin(\omega_2 t)e^{-z\sqrt{Re}}+\frac{2Re}{\pi}\sin(\omega_2 t)\times\int_0^\infty\frac{\eta^2\cos(z\eta)}{(\eta^2+Re)^2}d\eta+\frac{2Re^2}{\pi}\int_0^\infty\int_0^t\frac{\eta^2\cos(z\eta)}{(\eta^2+Re)^3}\sin(\omega_2 s)\times\exp\left[-\frac{\eta^2(t-s)}{\eta^2+Re}\right]ds\,d\eta-\frac{2}{\pi}\int_0^\infty\int_0^t\frac{\cos(\eta z)e^{-y\sqrt{\eta^2+Re}}}{\eta^2+Re}\times\left[\omega_1\sqrt{\eta^2+Re}\cos(\omega_1 s)+\frac{\eta^2}{\sqrt{\eta^2+Re}}\sin(\omega_1 s)\right]\times I_o\left[\frac{Re(t-s)}{2(\eta^2+Re)}\right]e^{-\frac{2\eta^2+Re}{2(\eta^2+Re)}(t-s)}d\eta ds-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma\cos(\eta z)$$

$$\times \left[\omega_1\sqrt{\eta^2+Re}\cos(\omega_1 s)+\frac{\eta^2}{\sqrt{\eta^2+Re}}\sin(\omega_1 s)\right] I_o\left[\frac{Re(\sigma-s)}{2(\eta^2+Re)}\right]$$
$$\times I_1\left[2\sqrt{uRe(t-\sigma)}\right] erfc\left(\frac{y}{2\sqrt{u}}\right)\sqrt{\frac{uRe}{t-\sigma}} \quad (49)$$
$$\times \exp\left[\frac{-2\eta^2+Re}{2(\eta^2+Re)}(\sigma-s)-u(\eta^2+Re)-(t-\sigma)\right] ds\, d\sigma\, du\, d\eta$$

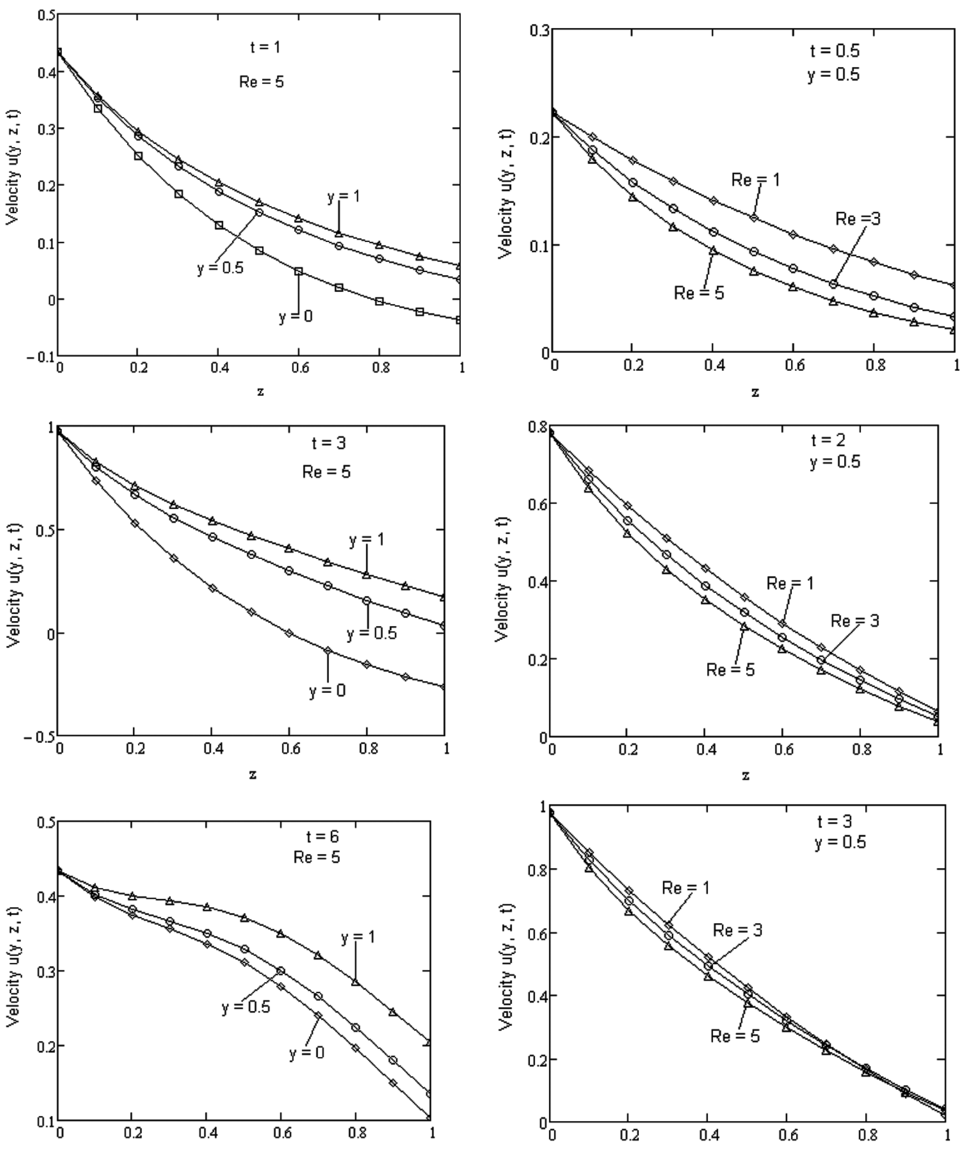

Figure 1. Profiles of the velocity $u(y, z, t)$ for since oscillations, $\omega_1 = \pi/5$. $\omega_2 = \pi/7$, and different values of y, Re, and t.

Numerical Results and Conclusions

In previous sections, exact analytical solutions have been established for a mixed initial boundary-value problem corresponding to the oscillating motion of a second-grade fluid along the inside of a rectangular edge. The motion of the fluid is due to the motion of the walls of an edge. One of them applies an oscillating shear to the fluid and the other one is oscillating in its plane. The solutions that have been obtained, in the form of simple and multiple integrals, satisfy all imposed initial and boundary conditions.

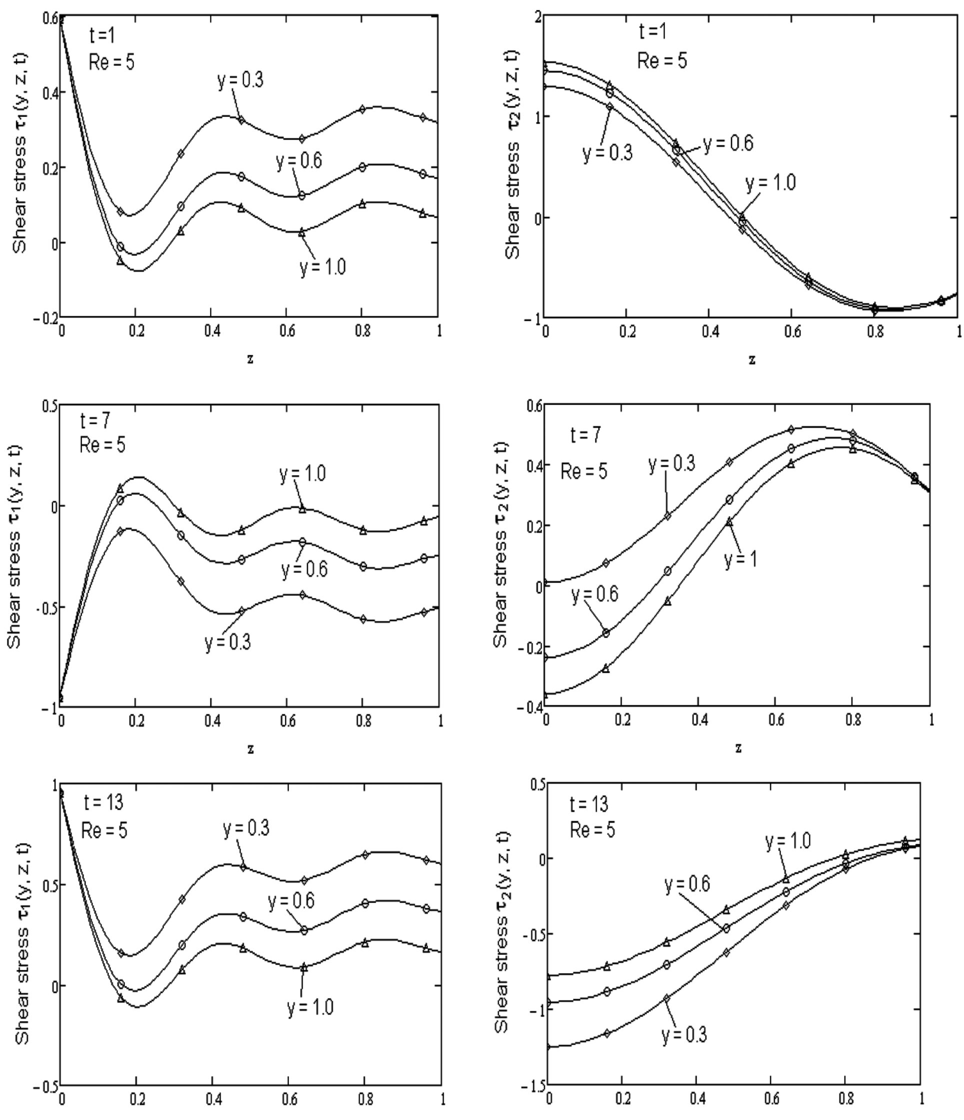

Figure 2. Profiles of shear stresses $\tau_1(y, z, t)$, $\tau_2(y, z, t)$ vs. z, for sine oscillations, $\omega_1 = \pi/5$, $\omega_2 = \pi/7$, and different values of y and t.

Finally, in order to reveal some relevant physical aspects of obtained results, the diagrams of the velocity $u(y, z, t)$ and the shear stresses $\tau_1(y, z, t)$ and $\tau_2(y, z, t)$ have been drawn against z for different values of y, Re, and time t (Figures 1 and 2) and against t for different values of y and z (Figures 3, 4, and 5). A series of diagrams were created for different situations with typical values. It clearly results from Figure 1 that, at the beginning of a cycle, the velocity of the fluid increases with respect to y and decreases with regard to the Reynolds number Re. Of course, it is a decreasing function with respect to z. The velocity of the fluid on the wall $z = 0$

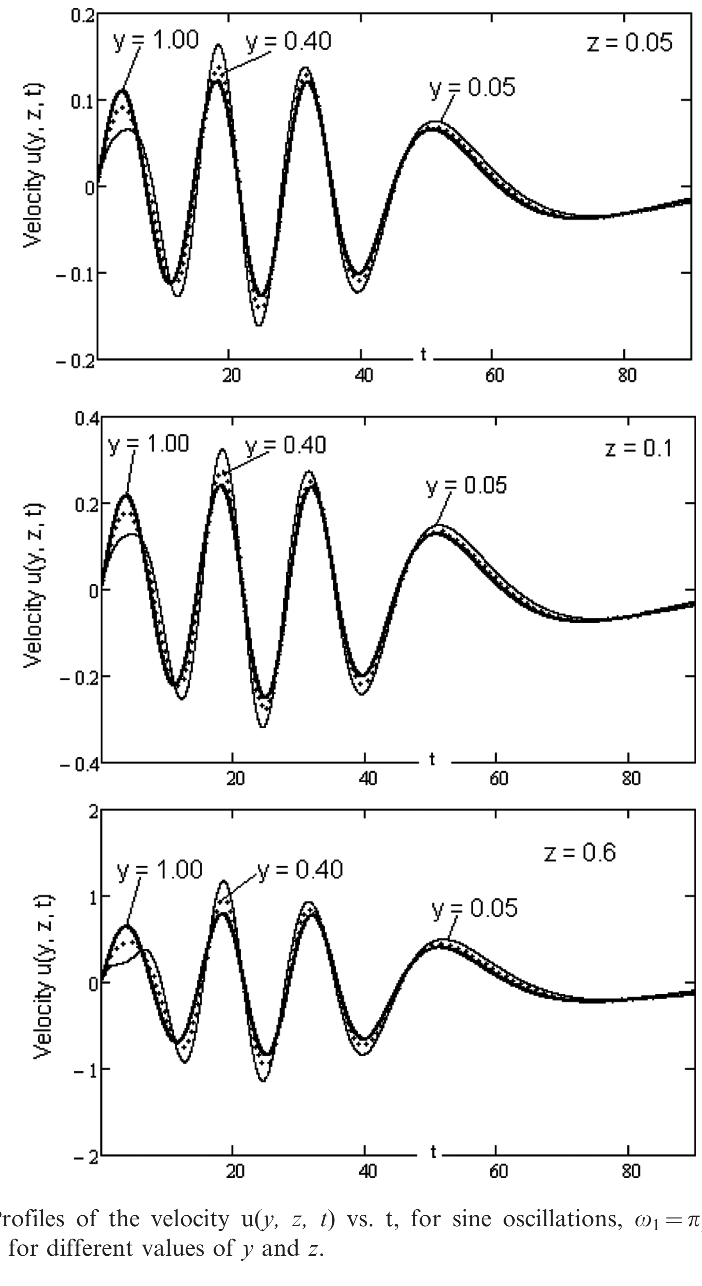

Figure 3. Profiles of the velocity u(y, z, t) vs. t, for sine oscillations, $\omega_1 = \pi/5$, $\omega_2 = \pi/7$, $Re = 3$, and for different values of y and z.

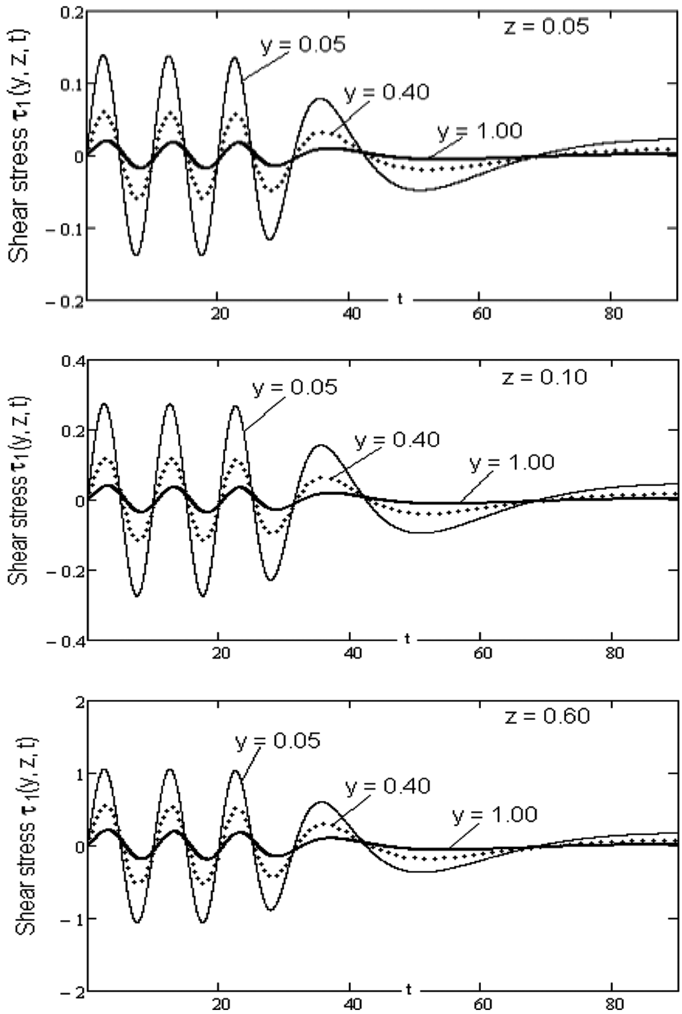

Figure 4. Profiles of the shear stress $\tau_1(y, z, t)$ vs. t. for sine oscillations, $\omega_1 = \pi/5$, $\omega_2 = \pi/7$, $Re = 3$, and for different values of y and z.

is same for a fixed value of Re and all values of y. It is different for different values of t, being just the velocity of the wall at that time. At small values of t, the velocity of the fluid decreases with respect to Re on the whole domain. After $t = 2$ the velocity of the fluid changes its monotony with respect to z. Figure 2 presents the diagrams of the shear stresses $\tau_1(y, z, t)$ and $\tau_2(y, z, t)$ for $t = 1$, 7, and 13, $Re = 5$, and $y = 0$, 0.3, 0.6, and 1. At the beginning of a cycle, more precisely for $t = 1$, the shear stress $\tau_1(y, z, t)$, as well as $\tau_2(y, z, t)$ in absolute value, decreases with respect to y. The shear stress $\tau_1(y, z, t)$ on the wall $z = 0$ in comparison with $\tau_2(y, z, t)$ has same value for all the values of y and the fixed values Re and t. Both stresses decrease with respect to z near the boundary, but their intervals of changing of the monotony are different. The oscillating values of $\tau_1(y, z, t)$ are clearly brought to light on the left side of Figure 2. The values of $\tau_2(y, z, t)$, as expected, are different for different values of y and the fixed values of Re and t.

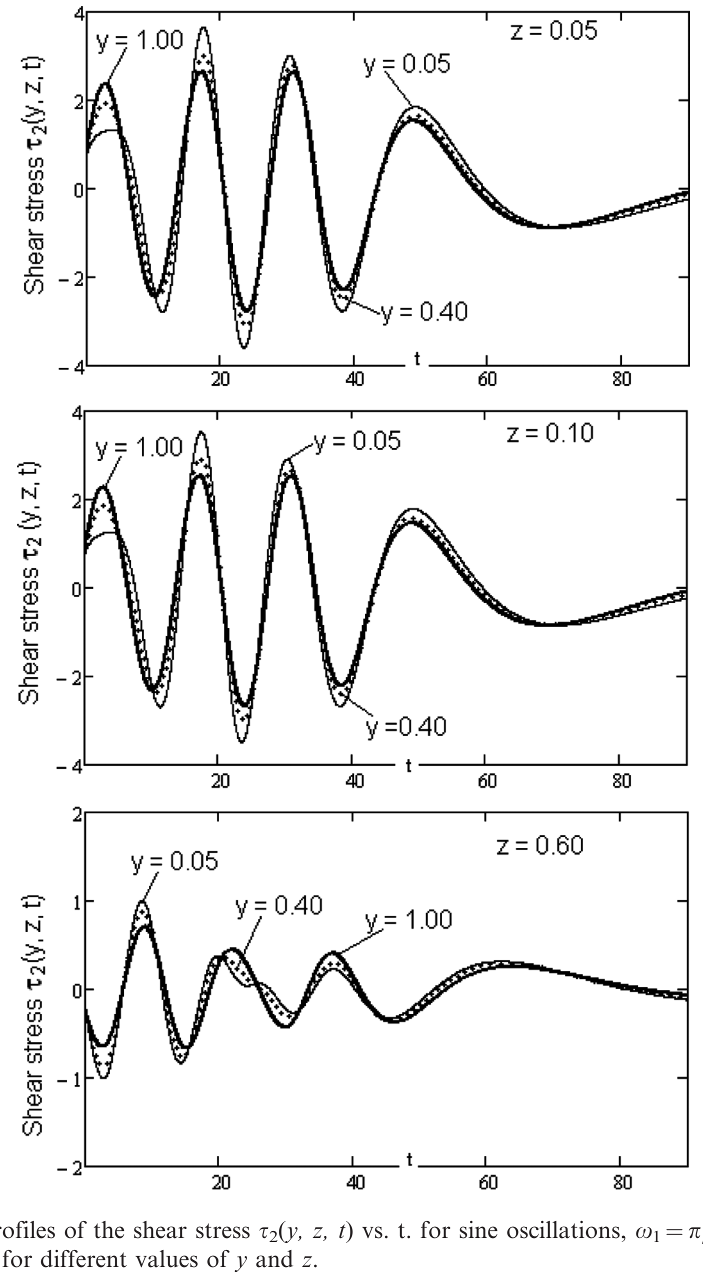

Figure 5. Profiles of the shear stress $\tau_2(y, z, t)$ vs. t. for sine oscillations, $\omega_1 = \pi/5$, $\omega_2 = \pi/t$, $Re = 3$, and for different values of y and z.

The variations of the velocity and the shear stresses with respect to time t are presented in Figures 3, 4, and 5 for different values of y and z. Qualitatively, the variation of the velocity is almost the same as that of the shear stress $\tau_2(y, z, t)$. Their amplitudes are increasing functions with respect to y. With respect to t, they are increasing functions on a small part of the interval $(0, \infty)$ and then decrease. The variation of $\tau_1(y, z, t)$ with respect to y seems to be the opposite. For all values of y and z, the oscillations of the three functions decrease in time. They are damping with time.

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Appendix

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{(q+\frac{2\eta^2+Re}{2(\eta^2+Re)})^2-(\frac{Re}{2(\eta^2+Re)})^2}}\right\}=I_o\left[\frac{Ret}{2(\eta^2+Re)}\right]e^{-\frac{2\eta^2+Re}{2(\eta^2+Re)}t} \qquad (A1)$$

$$\mathcal{L}^{-1}\{F(q)\}=\mathcal{L}^{-1}\left\{\frac{e^{-y\sqrt{q}}}{q}\right\}=f(t)=erfc\left(\frac{y}{2\sqrt{t}}\right) \qquad (A2)$$

If

$$\begin{aligned} f(t)&=\mathcal{L}^{-1}\{F(q)\} \quad \text{then} \\ \mathcal{L}^{-1}\{F[W(q)]\}&=\int_0^\infty f(u)g(u,t)du \quad \text{where} \\ g(u,t)&=\mathcal{L}^{-1}\left\{e^{-uW(\eta,q)}\right\} \end{aligned} \qquad (A3)$$

$$\mathcal{L}^{-1}\left\{e^{\frac{u}{q}}-1\right\}=\sqrt{\frac{u}{t}}I_1\left(2\sqrt{ut}\right) \qquad (A4)$$

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-y\sqrt{q}}}{q}\right\}=erf\left(\frac{y}{2\sqrt{u}}\right) \qquad (A5)$$

$$\int_0^\infty erfc(\frac{y}{2\sqrt{u}})e^{-u(\eta^2+Re)}du=\frac{e^{-y\sqrt{\eta^2+Re}}}{\eta^2+Re} \qquad (A6)$$

$$\int_0^\infty \frac{\eta\sin(z\eta)}{\eta^2+a^2}d\eta=\frac{\pi}{2}e^{-za}; \quad Re(a)\geq 0 \qquad (A7)$$

$$\int_0^\infty \frac{\eta\sin(z\eta)}{(\eta^2+b^2)^2+c^2}d\eta=\frac{\pi}{2|c|}e^{-Bz}\sin(Az) \qquad (A8)$$

$$\int_0^\infty \frac{\eta^3\sin(z\eta)}{(\eta^2+b^2)^2+c^2}d\eta=\frac{\pi}{2|c|}e^{-Bz}[|c|\cos(Az)-b^2\sin(Az)] \qquad (A9)$$

where $2A^2=\sqrt{b^4+c^2}-b^2$, $2B^2=\sqrt{b^4+c^2}+b^2$, $b^2=\frac{Re\omega_2^2}{1+\omega_2^2}$, $c^2=\frac{Re^2\omega_2^2}{(1+\omega_2^2)^2}$, $b\epsilon R$, $c\epsilon R^*$

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