Flow of a fractional Oldroyd-B fluid over a plane wall that applies a time-dependent shear to the fluid

Constantin Fetecau[a*], Nazish Shahid[b], Masood Khan[c]

[a] *Department of Mathematics, Technical University of Iasi, Iasi 700050, Associate member of Academy of Romanian Scientists, 050094 Bucuresti, Romania*

[b] *Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan*

[c] *Department of Mathematics, Quaid-i-Azam Universiry, Islamabad 44000,Pakistan*

Abstract

The unsteady flow of an incompressible Oldroyd-B fluid with fractional derivatives induced by a plane wall that applies a time-dependent shear stress $\mathrm{f}t^{a}$ to the fluid is studied using Fourier sine and Laplace transforms. Exact solutions for velocity and shear stress distributions are found in integral and series form in terms of generalized G functions. They are presented as a sum between the corresponding Newtonian solutions and non-Newtonian contributions and reduce to Newtonian solutions if relaxation and retardation times tend to zero. The solutions for fractional second grade and Maxwell fluids, as well as those for ordinary fluids, are obtained as limiting cases of general solutions. Finally, some special cases are considered and known solutions from the literature are recovered. An important relation with the first problem of Stokes is brought to light. The influence of fractional parameters on the fluid motion, as well as a comparison between models, is graphically illustrated.

Keywords: Fractional Oldroyd-B fluid, Time-dependent shear stress, Exact solutions.

1 Introduction

Many models have been proposed to describe the response characteristics of fluids that cannot be described by classical Navier-Stokes equations. Among them, the Oldroyd-B model can describe stress-relaxation, creep and normal stress differences that develop during simple

*Corresponding author E-mail address: *c_fetecau@yahoo.com*

shear flows. This model can be viewed as one of the most successful models for describing the response of a sub-class of polymeric liquids. It is amenable to analysis and more importantly experimental corroboration. An Oldroyd-B fluid is one which stores energy like a linearized elastic solid, its dissipation however being due to two dissipative mechanisms that implies that it arises from a mixture of two viscous fluids. Recently, there has been considerable interest in describing the behavior of incompressible Oldroyd-B fluids [1-11]. However, since one may expect that the behavior of viscoelastic liquids to deviate most from that of non-elastic non-Newtonian fluids in transient flows, it seems necessary to investigate new transient flows in order to have an overall view of elastic liquid behavior.

In view of the above motivation, we are interested to find exact solutions for the motion of an Oldroyd-B fluid induced by an infinite flat plate that applies a time-dependent shear stress to the fluid. Such exact solutions serve a dual purpose, that of providing an explicit solution to a problem that has physical relevance and as a means for testing the efficiency of complex numerical schemes for flows in more complicated flow domains. An interesting aspect of the problem to be studied is that unlike the usual no slip boundary condition, a boundary condition on the shear stress is used. This is very important as in some problems, what is specified is the force applied on the boundary. It is also important to bear in mind that the "no slip" boundary condition may not be necessarily applicable to flows of polymeric fluids that can slip or slide on the boundary. Thus, the shear stress boundary condition is particularly meaningful. Furthermore, in order to include a larger class of fluids, the general solutions will be established for Oldroyd-B fluids with fractional derivatives. Particularly, the solutions for Oldroyd-B fluids will be obtained as limiting cases.

In the last time, the fractional calculus is increasingly seen as an efficient tool and suitable framework within which useful generalizations of various classical physical concepts can be obtained. The list of its applications is quite long and augments almost yearly. It includes fractal media [12], fractional wave diffusion [13], fractional Hamiltonian dynamics [14,15] as well as many other topics in physics. In other cases, it has been shown that the constitutive equations employing fractional derivatives are linked to molecular theories [16]. In particular, it has been shown that the predictions of fractional derivative Maxwell model are in excellent agreement with the linear viscoelastic data in glass transition and α- relaxation zones [17]. The use of fractional derivatives within the context of viscoelasticity was firstly proposed by Germant [18]. Then, Slonimsky [19] introduced fractional derivatives into Kelvin-Voigt model to describe the relaxation processes. Subsequently, Bagley and Torvik [20, 21] and Koeller [22], among others, extended the theory. They demonstrated that the theory of viscoelasticity of coiling polymers and the theory of hereditary solid mechanics predict constitutive relations with fractional derivatives. As such, these models are consistent with basic theories and are not arbitrary constructions that happen to describe experimental data.

It is also important pointing out that the interest for viscoelastic fluids with fractional derivatives came from practical problems. In order to predict the dynamic response of viscous dampers, for instance, Makris et al. [23] firstly used conventional models of viscoelasticity. It was not possible to achieve satisfactory fit of the experimental data over the entire range of frequencies. However, a very good fit of the experimental data was achieved when the fractional Maxwell model

$$\tau + \lambda D^{\alpha}\tau = \mu D^{\beta}\gamma,$$

has been used. Here τ and γ are the shear stress and strain, λ and μ are generalized material constants and D^{α} is a fractional derivative operator of order α with respect to time. This model is a special case of the more general model of Bagley and Torvik [20]. It collapses to the conventional Maxwell model with $\alpha = \beta = 1$, in which case λ and μ become the relaxation time and the dynamic viscosity, respectively. Based on the fact that at vanishingly small strain rates, the behavior of the viscoelastic fluid reduces to that of Newtonian fluid, the parameter β was set equal to unity. The other three parameters were determined for the silicon gel fluid and the predicted mechanical properties are in excellent agreements with experimental results. Similar excellent agreements between frequency sweep experimental data obtained on other polymers (e.g. polystyrenes) and theoretical predictions of linear fractional derivative models are reported in [24-26].

However, despite these successful attempts, it must be emphasized that a constitutive relation should be expressed in a three dimensional setting such that it is also frame indifferent. The first objective law which characterizes an incompressible fractional derivative Maxwell fluid seems to be that of Palade et al. [27, Eq. (16)]. This constitutive relation, under linearization, reduces to the fractional integral Maxwell model exhibited in [27. Eq. (8)]. Using the definition of a fractional integral, the last equality (8) is equivalent to the present equality proposed by Makris et al. [23]. Consequently, if one wishes to study one-dimensional behavior only, then it would appear that these models are successful. So in the following we shall establish exact solutions for velocity and shear stress corresponding to the unsteady flow of an incompressible fractional Oldroyd-B fluid due to an infinite plate that applies a time-dependent shear to the fluid. These solutions, that satisfy all imposed initial and boundary conditions, are presented as sums of Newtonian solutions and non-Newtonian contributions. They can easily be specialized to give both the solutions for fractional Maxwell and second grade fluids and those for ordinary fluids. Finally, the influence of fractional parameters on the fluid motion, as well as a comparison between models, is underlined by graphical illustrations.

2 Governing equations

For the problem under consideration we shall assume the velocity field $\mathbf{v}$ and the extra-stress tensor $\mathbf{S}$ of the form

$$\mathbf{v} = \mathbf{v}(y,t) = v(y,t)\mathbf{i},\ \ \mathbf{S} = \mathbf{S}(y,t), \tag{1}$$

where $\mathbf{i}$ is the unit vector along the x-direction of the Cartesian coordinate system x, y and z. For this flow, the constraint of incompressibility is automatically satisfied. Substituting Eq. (1) into the constitutive equations corresponding to an incompressible Oldroyd-B fluid and assuming that the fluid is at rest till the moment $t = 0$, we obtain the relevant equation [2,6]

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(y,t) = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\frac{\partial v(y,t)}{\partial y}, \tag{2}$$

where μ is the viscosity of the fluid, λ and λ_r are relaxation and retardation times and $\tau(y,t) = S_{xy}(y,t)$ is the nontrivial shear stress.

In the absence of body forces and a pressure gradient in the flow direction, the balance of linear momentum leads to the significant equation

$$\frac{\partial\tau(y,t)}{\partial y} = \rho\frac{\partial v(y,t)}{\partial t}, \tag{3}$$

where ρ is the density of the fluid. Eliminating $\tau(y,t)$ between Eqs. (2) and (3), we obtain the following governing equation

$$(1 + \lambda\frac{\partial}{\partial t})\frac{\partial v(y,t)}{\partial t} = \nu(1 + \lambda_r\frac{\partial}{\partial t})\frac{\partial^2 v(y,t)}{\partial y^2}, \tag{4}$$

for velocity. Here, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid.

The governing equations corresponding to incompressible fractional Oldroyd-B fluids (FOF), in such motions [28,29]

$$(1 + \lambda^\alpha D_t^\alpha)\,\tau(y,t) = \mu\left(1 + \lambda_r^\beta D_t^\beta\right)\frac{\partial v(y,t)}{\partial y}, \tag{5}$$

$$(1 + \lambda^\alpha D_t^\alpha)\frac{\partial v(y,t)}{\partial t} = \nu(1 + \lambda_r^\beta D_t^\beta)\frac{\partial^2 v(y,t)}{\partial y^2}, \tag{6}$$

are derived from Eqs. (2) and (4) via substituting the inner time derivatives by the fractional differential operator (also called Caputo fractional operator with zero initial condition) [30, 31]

$$D_t^p f(t) = \frac{1}{\Gamma(1-p)}\int_0^t \frac{f'(\tau)}{(t-\tau)^p}d\tau;\quad 0 \le p < 1, \tag{7}$$

where $\Gamma(.)$ is the Gamma function. The constraint $\alpha \geq \beta$ is explained in [32]. In the following the system of fractional partial differential equations (5) and (6) with appropriate initial and boundary conditions will be solved by means of integral transforms.

3 Statement of the problem and its solution

Let us consider an incompressible FOF occupying the space above a flat plate situated in the (x,z) plane. Initially, the fluid as well as the plate is at rest. At time $t = 0^+$ let the plate be pulled with the time-dependent shear

$$\tau(0,t) = \frac{f}{\lambda^\alpha}\int_0^t (t-s)^a G_{\alpha,0,1}\left(-\frac{1}{\lambda^\alpha}, s\right) ds; \quad a \geq 0, \tag{8}$$

along the x-axis. Here, f and a are constants and (see [33], the pages 14 and 15)

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty} \frac{\Gamma(c+j)}{\Gamma(j+1)\Gamma(c)} \frac{t^{a(j+c)-b-1}}{\Gamma[a(j+c)-b]} d^j.$$

Owing to the shear the fluid is gradually moved. Its velocity is of the form $(1)_1$, the governing equations are given by Eqs. (5) and (6) while the appropriate initial and boundry conditions are

$$v(y,0) = \frac{\partial v(y,0)}{\partial t} = 0, \;\; \tau(y,0) = 0; \quad y > 0, \tag{9}$$

$$(1+\lambda^\alpha D_t^\alpha)\,\tau(0,t) = \mu\left(1+\lambda_r^\beta D_t^\beta\right)\left.\frac{\partial v(y,t)}{\partial y}\right|_{y=0} = f t^a; \;\; t > 0. \tag{10}$$

Moreover, the natural condition

$$v(y,t) \to 0 \;\; as \;\; y \to \infty, \tag{11}$$

has to be also satisfied. Of course, as we shall see later, $\tau(0,t)$ given by Eq. (8) is just the solution of the fractional differential equation $(10)_1$.

3.1 Calculation of the velocity

Multiplying Eq. (6) by $\sqrt{\frac{2}{\pi}} cos(y\xi)$, integrating the result with respect to y from 0 to infinity and taking into account the above initial and boundary conditions, we find that

$$(1+\lambda^\alpha D_t^\alpha)\frac{\partial v_c(\xi,t)}{\partial t} + \nu\xi^2\left(1+\lambda_r^\beta D_t^\beta\right) v_c(\xi,t) = -\frac{f}{\rho} t^a \sqrt{\frac{2}{\pi}}; \;\; \xi, t > 0, \tag{12}$$

where the Fourier cosine transform $v_c(\xi, t)$ of $v(y, t)$ has to satisfy the initial conditions

$$v_c(\xi, 0) = \frac{\partial v_c(\xi, 0)}{\partial t} = 0; \;\; \xi > 0. \tag{13}$$

Applying the Laplace transform to Eq. (12), using the Laplace transform formula for sequential fractional derivatives [31] and having in mind the initial conditions (13), we find for the image function $\bar{v}_c(\xi, q)$ of $v_c(\xi, t)$ the expression

$$\bar{v}_c(\xi, q) = -\frac{f}{\rho}\sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{q + \lambda^\alpha q^{\alpha+1} + \nu\xi^2 + \gamma\xi^2 q^\beta}, \tag{14}$$

where q is the transform parameter and $\gamma = \nu\lambda_r^\beta$. In order to obtain $v_c(\xi, t) = L^{-1}\{\bar{v}_c(\xi, q)\}$ and to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method [34]. However in order to obtain a more suitable presentation of final results, we firstly rewrite Eq. (14) in the equivalent form

$$\bar{v}_c(\xi, q) = -\frac{f}{\mu}\sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{\xi^2} + \frac{f}{\mu}\sqrt{\frac{2}{\pi}}\frac{1}{\xi^2}F(\xi, q) + \frac{f}{\rho}\sqrt{\frac{2}{\pi}}F(\xi, q)G(\xi, q), \tag{15}$$

where $F(\xi, q) = F_1(q)F_2(\xi, q)$ and

$$F_1(q) = \frac{\Gamma(a+1)}{q^a}, \;\; F_2(\xi, q) = \frac{1}{q + \nu\xi^2} \;\;\; \textit{and} \;\;\; G(\xi, q) = \frac{\lambda^\alpha q^\alpha + \gamma\xi^2 q^{\beta-1}}{q + \lambda^\alpha q^{\alpha+1} + \nu\xi^2 + \gamma\xi^2 q^\beta}. \tag{16}$$

Denoting by $f_1(t)$, $f_2(\xi, t)$, $f(\xi, t)$ and $g(\xi, t)$ the inverse Laplace transforms of $F_1(q)$, $F_2(\xi, q)$, $F(\xi, q)$ and $G(\xi, q)$ and bearing in mind Eq. (A_1) from the Appendix A, we can write

$$v_c(\xi, t) = -\frac{f}{\mu}\sqrt{\frac{2}{\pi}}\frac{t^a}{\xi^2} + \frac{f}{\mu}\sqrt{\frac{2}{\pi}}\frac{1}{\xi^2}f(\xi, t) + \frac{f}{\rho}\sqrt{\frac{2}{\pi}}h(\xi, t), \tag{17}$$

where

$$f(\xi, t) = (f_1 * f_2)(t) = \begin{cases} e^{-\nu\xi^2 t}, & a = 0 \\ a\int_0^t (t-s)^{a-1}e^{-\nu\xi^2 s}ds, & a > 0 \end{cases} \tag{18}$$

and $h(\xi, t) = L^{-1}\{F(\xi, q)G(\xi, q)\} = (f * g)(t) = \int_0^t f(\xi, t-s)g(\xi, s)ds$.

Applying the inverse Fourier transform to Eq. (17) and using Eqs. (A_2) and (A_3), we find for the velocity $v(y, t)$, the simple expression

$$v(y, t) = v_N(y, t) + \frac{2f}{\rho\pi}\int_0^\infty h(\xi, t)\cos(y\xi)d\xi, \tag{19}$$

where [34, Eq. (4.1) with α=0]

$$v_N(y,t)=\frac{f}{\mu}yt^a-\frac{2f}{\mu\pi}\int_0^\infty\{t^a-f(\xi,t)\cos(y\xi)\}\frac{d\xi}{\xi^2}, \tag{20}$$

is the velocity corresponding to a Newtonian fluid performing the same motion and

$$\begin{aligned} h(\xi,t)=&\sum_{k=0}^{\infty}\sum_{m=0}^{k}\left(-\frac{\nu\xi^2}{\lambda^\alpha}\right)^k\frac{k!\lambda_r^{m\beta}}{m!(k-m)!}\\ \times\int_0^t f(\xi,t-s)&\left\{G_{\alpha,\alpha_m,k+1}\left(-\frac{1}{\lambda^\alpha},s\right)+\nu\xi^2\frac{\lambda_r^\beta}{\lambda^\alpha}G_{\alpha,\beta_m,k+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right\}ds, \end{aligned} \tag{21}$$

with $\alpha_m=\alpha+m\beta-k-1$ and $\beta_m=(1+m)\beta-k-2$.

The velocity $v(y,t)$, as it results from Eq. (19), is presented as a sum between the Newtonian solution $v_N(y,t)$ and the non-Newtonian contribution

$$v_{nN}(y,t)=\frac{2f}{\rho\pi}\int_0^\infty h(\xi,t)\cos(y\xi)d\xi. \tag{22}$$

Of course, in view of Eq. (A_4), it clearly results that for λ_r *and* $\lambda\to 0$, $v_{nN}(y,t)\to 0$ and therefore $v(y,t)\to v_N(y,t)$.

3.2 Calculation of the shear stress

Applying the Laplace transform to Eq. (5), we obtain

$$\bar{\tau}(y,q)=\mu\frac{1+\lambda_r^\beta q^\beta}{1+\lambda^\alpha q^\alpha}\frac{\partial\bar{v}(y,q)}{\partial y}, \tag{23}$$

where

$$\bar{v}(y,q)=-\frac{2f}{\rho\pi}\frac{\Gamma(a+1)}{q^{a+1}}\int_0^\infty\frac{cos(y\xi)}{q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2+\gamma\xi^2q^\beta}d\xi, \tag{24}$$

is obtained from Eq. (14). Introducing Eq. (24) in Eq. (23), it results

$$\bar{\tau}(y,q)=\frac{2f\nu}{\pi}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1+\lambda_r^\beta q^\beta}{1+\lambda^\alpha q^\alpha}\int_0^\infty\frac{\xi\sin(y\xi)}{q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2+\gamma\xi^2q^\beta}d\xi. \tag{25}$$

In the following, in order to obtain for the shear stress $\tau(y,t)=L^{-1}\{\tau(y,q)\}$ a similar form to that of velocity, we shall use the identity

$$\frac{\Gamma(a+1)}{q^{a+1}}\frac{1+\lambda_r^\beta q^\beta}{1+\lambda^\alpha q^\alpha}\frac{1}{q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2+\gamma\xi^2q^\beta}=\frac{1}{\nu\xi^2}\left[\frac{\Gamma(a+1)}{q^{a+1}}-F(\xi,q)\right]+F_1(q)G_1(\xi,q)G_2(\xi,q),$$

where $F_1(.)$ and $F(\xi,.)$ have been previously defined,

$$G_1(\xi,q)=\frac{1}{q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2+\nu\xi^2\lambda^\alpha q^\alpha},\ G_2(\xi,q)=\frac{\lambda_r^\beta q^\beta-2\lambda^\alpha q^\alpha-\lambda^{2\alpha}q^{2\alpha}-\gamma\xi^2\lambda^\alpha q^{\alpha+\beta-1}-\nu\xi^2\lambda^\alpha q^{\alpha-1}}{q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2+\gamma\xi^2 q^\beta}$$

and follow the same way as before. In order to avoid repetition, we give the final result in the simple form

$$\tau(y,t)=\tau_N(y,t)+\frac{2f\nu}{\pi}\int_0^\infty \xi\sin(y\xi)(f_1*g_1*g_2)(t)d\xi, \tag{26}$$

where [34, Eq. (4.2) with α=0]

$$\tau_N(y,t)=ft^a-\frac{2f}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi}f(\xi,t)d\xi, \tag{27}$$

represents the shear stress corresponding to Newtonian fluids

$$f_1(t)=\begin{cases}\delta(t), & a=0\\ at^{a-1}, & a>0\end{cases},\delta(.)\ \textit{being the Dirac delta function},$$

$$g_1(\xi,t)=\frac{1}{\lambda^\alpha}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\left(-\frac{\nu\xi^2}{\lambda^\alpha}\right)^k\frac{k!\lambda^{m\alpha}}{m!(k-m)!}G_{\alpha,m\alpha-k-1,k+1}\left(-\frac{1}{\lambda^\alpha},t\right),$$

$$g_2(\xi,t)=\frac{1}{\lambda^\alpha}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\left(-\frac{\nu\xi^2}{\lambda^\alpha}\right)^k\frac{k!\lambda_r^{m\beta}}{m!(k-m)!}\begin{bmatrix}\lambda_r^\beta G_{\alpha,\beta_m+1,k+1}\left(-\frac{1}{\lambda^\alpha},t\right)-2\lambda^\alpha G_{\alpha,\alpha_m,k+1}\left(-\frac{1}{\lambda^\alpha},t\right)\\ -\lambda^{2\alpha}G_{\alpha,\alpha_m+\alpha,k+1}\left(-\frac{1}{\lambda^\alpha},t\right)-\gamma\xi^2\lambda^\alpha G_{\alpha,\beta_m+\alpha,k+1}\left(-\frac{1}{\lambda^\alpha},t\right)\\ -\nu\xi^2\lambda^\alpha G_{\alpha,\alpha_m-1,k+1}\left(-\frac{1}{\lambda^\alpha},t\right)\end{bmatrix}.$$

A simple analysis clearly shows that $\tau(y,t)\to\tau_N(y,t)$ for λ_r and $\lambda\to 0$.

3.3 Special cases $a=0,1,2,3,...$

By making $a=0$ into Eqs. (19) and (26) and having in mind Eq. (18), (A_5) and the entry 6 of Table 5 from [35], we find that

$$v_0(y,t)=v_{0N}(y,t)+\frac{2f}{\rho\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^{m\beta}}{m!(k-m)!}\int_0^\infty\left(-\frac{\nu\xi^2}{\lambda^\alpha}\right)^k\cos(y\xi)$$
$$\times\int_0^t e^{-\nu\xi^2(t-s)}\left\{G_{\alpha,\alpha_m,k+1}\left(-\frac{1}{\lambda^\alpha},s\right)+\nu\xi^2\frac{\lambda_r^\beta}{\lambda^\alpha}G_{\alpha,\beta_m,k+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right\}dsd\xi, \tag{28}$$

$$\tau_0(y,t)=\tau_{0N}(y,t)+\frac{2f\nu}{\pi}\int_0^\infty \xi\sin(y\xi)\int_0^t g_1(\xi,t-s)g_2(\xi,s)dsd\xi, \tag{29}$$

where the expressions of

$$v_{0N}(y,t)=\frac{f}{\mu}y-\frac{2f}{\mu\pi}\int_0^\infty\{1-e^{-\nu\xi^2 t}\cos(y\xi)\}\frac{d\xi}{\xi^2}=\frac{fy}{\mu}erfc\left(\frac{y}{2\sqrt{\nu t}}\right)-\frac{2f}{\mu}\sqrt{\frac{\nu t}{\pi}}exp\left(-\frac{y^2}{4\nu t}\right) \tag{30}$$

and

$$\tau_{0N}(y,t)=f-\frac{2f}{\pi}\int_0^t\frac{\sin(y\xi)}{\xi}e^{-\nu\xi^2 t}d\xi=f\ erfc\left(\frac{y}{2\sqrt{\nu t}}\right), \tag{31}$$

are identical to those obtained in [3, Eqs. (4.3) and (4.4)].

The solutions corresponding to $a=1$, namely,

$$v_1(y,t)=v_{1N}(y,t)+\frac{2f}{\rho\pi}\sum_{k=0}^\infty\sum_{m=0}^k\frac{k!\lambda_r^{m\beta}}{m!(m-k)!}\int_0^\infty\left(-\frac{\nu\xi^2}{\lambda^\alpha}\right)^k\cos(y\xi)$$
$$\times\int_0^t e^{-\nu\xi^2(t-s)}\left\{G_{\alpha,\alpha_m-1,k+1}\left(-\frac{1}{\lambda^\alpha},s\right)+\nu\xi^2\frac{\lambda_r^\beta}{\lambda^\alpha}G_{\alpha,\beta_m-1,k+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right\}dsd\xi, \tag{32}$$

$$\tau_1(y,t)=\tau_{1N}(y,t)+\frac{2f\nu}{\pi}\int_0^\infty\xi\sin(y\xi)\int_0^t(g_1*g_2)(s)dsd\xi, \tag{33}$$

where

$$v_{1N}(y,t)=\frac{f}{\mu}yt-\frac{2f}{\mu\pi}\int_0^\infty\left\{t-\frac{1-e^{-\nu\xi^2 t}}{\nu\xi^2}\cos(y\xi)\right\}\frac{d\xi}{\xi^2}, \tag{34}$$

$$\tau_{1N}(y,t)=ft-\frac{2f}{\pi}\int_0^\infty\frac{1-e^{-\nu\xi^2 t}}{\xi^3}\sin(y\xi)d\xi=f\int_0^t erfc\left(\frac{y}{2\sqrt{\nu s}}\right)ds, \tag{35}$$

are also identical to those obtained in [36, Eqs. (21) and (22)] by a different technique. In order to get Eq. (32), for instance, we made an integration by parts into Eq. (21) and used Eqs. (18) and (A_6). A simple analysis shows that

$$v_1(y,t)=\int_0^t v_0(y,s)ds\quad and\quad \tau_1(y,t)=\int_0^t\tau_0(y,s)ds. \tag{36}$$

Lengthy but straightforward computations allow us to prove that

$$v_n(y,t)=n!\int_0^t\int_0^{s_1}\int_0^{s_2}...\int_0^{s_{n-1}}v_0(y,s_n)ds_nds_{n-1}...ds_1, \tag{37}$$

$$\tau_n(y,t) = n! \int_0^t \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_{n-1}} \tau_0(y,s_n) ds_n ds_{n-1} ... ds_1. \tag{38}$$

4 Limiting cases

4.1 The case $\lambda_r \to 0$ (fractional Maxwell fluids)

Making $\lambda_r \to 0$ into Eqs. (19) and (26) we get the solutions

$$v_{FM}(y,t) = v_N(y,t) + \frac{2f}{\rho\pi} \int_0^\infty \cos(y\xi) h_{FM}(\xi,t) d\xi, \tag{39}$$

$$\tau_{FM}(y,t) = \tau_N(y,t) + \frac{2f\nu}{\pi} \int_0^\infty \xi \sin(y\xi)(f_1 * g_1 * g_{2FM})(t) d\xi, \tag{40}$$

corresponding to a Maxwell fluid with fractional derivatives performing the same motion. Here $f_1(.)$ and $g_1(\xi,.)$ are the same as before and

$$h_{FM}(\xi,t) = \sum_{k=0}^\infty \left(-\frac{\nu\xi^2}{\lambda^\alpha}\right)^k \int_0^t f(\xi,t-s) G_{\alpha,\alpha-k-1,k+1}\left(-\frac{1}{\lambda^\alpha},s\right) ds,$$

$$g_{2FM}(\xi,t) = -\sum_{k=0}^\infty \left(-\frac{\nu\xi^2}{\lambda^\alpha}\right)^k \left\{ 2G_{\alpha,\alpha-k-1}\left(-\frac{1}{\lambda^\alpha},t\right) + \lambda^\alpha G_{\alpha,2\alpha-k-1}(-\frac{1}{\lambda^\alpha},t) + \nu\xi^2 G_{\alpha,\alpha-k-2,k+1}\left(-\frac{1}{\lambda^\alpha},t\right)\right\}.$$

Of course, in view of Eq. (A_4), $v_{FM}(y,t) \to v_N(y,t)$ and $\tau_{FM}(y,t) \to \tau_N(y,t)$ if $\lambda \to 0$.

4.2 The case $\lambda \to 0$ (fractional second grade fluids)

The solutions corresponding to second grade fluids with fractional derivatives can also be obtained as limiting cases of general solutions using Eq. (A_4). However, simpler but equivalent forms of these solutions, namely

$$v_{FSG}(y,t) = v_N(y,t) + \frac{2f}{\rho\pi}\gamma \int_0^\infty \xi^2 \cos(y\xi) h_{FSG}(\xi,t) d\xi, \tag{41}$$

$$\tau_{FSG}(y,t) = \tau_N(y,t) + \frac{2f}{\pi}\gamma \int_0^\infty \xi \sin(y\xi) g_{FSG}(\xi,t) d\xi, \tag{42}$$

are obtained making $\lambda \to 0$ into Eqs. (14) and (25) and using the identity

$$\frac{1}{q+\nu\xi^2+\gamma\xi^2 q^\beta} = \sum_{k=0}^\infty (-\gamma\xi^2)^k \frac{q^{\beta k}}{(q+\nu\xi^2)^{k+1}}.$$

The two functions from Eqs. (41) and (42) are given by

$$h_{FSG}(\xi,t)=\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\int_0^t f(\xi,t-s)G_{1,\beta k+\beta-1,k+1}(-\nu\xi^2,s)ds,$$

$$g_{FSG}(\xi,t)=\Gamma(a+1)\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\int_0^t e^{-\nu\xi^2(t-s)}G_{1,\beta(k+1)-a,k+1}(-\nu\xi^2,s)ds.$$

4.3 The case $\alpha=\beta=1$ (Oldroyd-B fluids)

By making $\alpha=\beta=1$ into Eqs. (19) and (26) we obtain the similar solutions for Oldroyd-B fluids. The velocity field $v_0(y,t)$, for instance, has the form

$$v_0(y,t)=v_N(y,t)+\frac{2f}{\rho\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\int_0^{\infty}\left(-\frac{\nu\xi^2}{\lambda}\right)^k\cos(y\xi)$$
$$\times\int_0^t f(\xi,t-s)\left\{G_{1,m-k,k+1}\left(-\frac{1}{\lambda},s\right)+\nu\xi^2\frac{\lambda_r}{\lambda}G_{1,m-k-1,k+1}\left(-\frac{1}{\lambda},s\right)\right\}dsd\xi, \tag{43}$$

where $f(\xi,t)$ is given by Eq. (18). For $a=0$, the corresponding solution

$$v_{00}(y,t)=v_N(y,t)+\frac{2f}{\rho\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!\lambda_r^m}{m!(k-m)!}\int_0^{\infty}\left(-\frac{\nu\xi^2}{\lambda}\right)^k\cos(y\xi)$$
$$\times\int_0^t e^{-\nu\xi^2(t-s)}\left\{G_{1,m-k,k+1}\left(-\frac{1}{\lambda},s\right)+\nu\xi^2\frac{\lambda_r}{\lambda}G_{1,m-k-1,k+1}\left(-\frac{1}{\lambda},s\right)\right\}dsd\xi, \tag{44}$$

as it results from Fig. 1, is equivalent to the known solution

$$v_{00}(y,t)=\frac{fy}{\mu}-\frac{2f}{\mu\pi}\int_0^{\infty}\left\{1-\frac{q_2exp(q_1t)-q_1exp(q_2t)}{q_2-q_1}\cos(y\xi)\right\}\frac{1}{\xi^2}d\xi, \tag{45}$$

obtained in [3] by a different technique. In the last relation q_1 and q_2 are the roots of the algebraic equation of second grade $\lambda q^2+(1+\gamma\xi^2)q+\nu\xi^2=0$.

Equivalent but simpler expressions for $v_0(y,t)$ and $\tau_0(y,t)$ can be obtained using new suitable decompositions for the corresponding functions $G(\xi,.)$, $G_1(\xi,.)$ and $G_2(\xi,.)$. Using the identity

$$\frac{\lambda q+\gamma\xi^2}{\lambda q^2+(1+\gamma\xi^2)q+\nu\xi^2}=\frac{q+\frac{\alpha}{2\lambda}}{(q+\frac{\alpha}{2\lambda})^2-(\frac{\beta}{2\lambda})^2}-\frac{1-\gamma\xi^2}{\beta}\frac{\frac{\beta}{2\lambda}}{(q+\frac{\alpha}{2\lambda})^2-(\frac{\beta}{2\lambda})^2},$$

for instance, we find that

$$v_0(y,t) = v_N(y,t) + \frac{2f}{\rho\pi}\int_0^\infty \cos(y\xi)\int_0^t f(\xi, t-s)\left[ch\left(\frac{\beta s}{2\lambda}\right) - \frac{1-\gamma\xi^2}{\beta} sh\left(\frac{\beta s}{2\lambda}\right)\right] exp\left(-\frac{\alpha s}{2\lambda}\right) ds d\xi, \tag{46}$$

where $\alpha = 1+\gamma\xi^2$ and $\beta = \sqrt{(1+\gamma\xi^2)^2 - 4\nu\lambda\xi^2}$. The equivalence of the solutions given by Eqs. (45) and (46) (with $a = 0$) is shown by Fig.2. It can also be proved by direct computations.

4.4 The case $\lambda_r \to 0$ and $\alpha = 1$ (Maxwell fluids)

The solutions corresponding to Maxwell fluids performing the same motion are immediately obtained from Eqs. (19) and (26) by making $\lambda_r \to 0$ and $\alpha = 1$. However, they can also be obtained from Eqs. (39) and (40) for $\alpha = 1$ or from the solutions corresponding to Oldroyd-B fluids for $\lambda_r \to 0$.

4.5 The case $\lambda \to 0$ and $\beta = 1$ (Second grade fluids)

By letting now $\beta = 1$ into Eqs. (41) and (42) we get the similar solutions for second grade fluids. They can also be obtained from general solutions (for $\lambda \to 0$ and $\beta = 1$) or from the solutions of the Oldroyd-B fluids (for $\lambda \to 0$). The solutions corresponding to $a = 0$ and 1, for instance,

$$v_{0SG}(y,t) = v_{0N}(y,t) + \frac{2f}{\rho\pi}\gamma\sum_{k=0}^\infty \int_0^\infty \xi^2(-\gamma\xi^2)^k \cos(y\xi)\int_0^t e^{-\nu\xi^2(t-s)} G_{1,k,k+1}(-\nu\xi^2, s) ds d\xi, \tag{47}$$

$$\tau_{0SG}(y,t) = \tau_{0N}(y,t) + \frac{2f}{\pi}\gamma\sum_{k=0}^\infty \int_0^\infty \xi(-\gamma\xi^2)^k \sin(y\xi)\int_0^t e^{-\nu\xi^2(t-s)} G_{1,k+1,k+1}(-\nu\xi^2, s) ds d\xi, \tag{48}$$

$$v_{1SG}(y,t) = v_{1N}(y,t) + \frac{2f}{\rho\pi}\gamma\sum_{k=0}^\infty \int_0^\infty \xi^2(-\gamma\xi^2)^k \cos(y\xi)\int_0^t e^{-\nu\xi^2(t-s)} G_{1,k-1,k+1}(-\nu\xi^2, s) ds d\xi, \tag{49}$$

$$\tau_{1SG}(y,t) = \tau_{1N}(y,t) + \frac{2f}{\pi}\gamma\sum_{k=0}^\infty \int_0^\infty \xi(-\gamma\xi^2)^k \sin(y\xi)\int_0^t e^{-\nu\xi^2(t-s)} G_{1,k,k+1}(-\nu\xi^2, s) ds d\xi, \tag{50}$$

are immediately obtained from Eqs. (41) and (42) for $a = 0$ or 1 and $\beta = 1$. Finally, it is worth pointing out that in view of Eqs. $(B_1), (B_2), (B_3)$ and (B_4) from the Appendix B, these solutions take the simplified forms

$$v_{0SG}(y,t) = \frac{f}{\mu}yt - \frac{2f}{\mu\pi}\int_0^\infty \left[1 - \cos(y\xi) exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right)\right]\frac{1}{\xi^2} d\xi, \tag{51}$$

$$\tau_{0SG}(y,t) = fH(t)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi(1+\gamma\xi^2)} exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right) d\xi\right], \tag{52}$$

and

$$v_{1SG}(y,t) = \frac{f}{\mu}yt - \frac{2f}{\mu\pi}\int_0^\infty \left\{t - \frac{1+\gamma\xi^2}{\nu\xi^2}\left[1 - exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right)\right]\cos(y\xi)\right\}\frac{1}{\xi^2}d\xi, \tag{53}$$

$$\tau_{1SG}(y,t) = ft - \frac{2f}{\nu\pi}\int_0^\infty \left[1 - exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right)\right]\frac{\sin(y\xi)}{\xi^3}d\xi, \tag{54}$$

obtained in [3, Eq. (4.1)], respectively [36, Eqs. (19) and (20)] by a different technique.

5 Conclusions and numerical results

The main purpose of this paper is to provide exact solutions for velocity and shear stress corresponding to the unsteady motion of an Oldroyd-B fluid due to an infinite plate that applies a shear stress ft^a to the fluid. However, for generality, these solutions have been established for a larger class of fluids, namely Oldroyd-B fluids with fractional derivatives. They are presented as a sum of Newtonian solutions and non-Newtonian contributions and satisfy all imposed initial and boundary conditions. The non-Newtonian contributions, as expected, tend to zero for λ and $\lambda_r \to 0$. Furthermore, the similar solutions for fractional Maxwell and second grade fluids as well as those for ordinary Oldroyd-B , Maxwell and second grade fluids are also obtained as limiting cases of general solutions for $\lambda_r \to 0$ or $\lambda \to 0$, respectively $\alpha = \beta = 1$, $\lambda_r \to 0$ and $\alpha = 1$ or $\lambda \to 0$ and $\beta = 1$.

Finally, in order to establish a relation with the motion over a moving plate, let us remember the velocity fields (see [37, Eq. (3)] and [6, Eq. (23)])

$$v_0(y,t) = VH(t)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi(1+\gamma\xi^2)} exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right) d\xi\right], \tag{55}$$

$$v_1(y,t) = At - \frac{2A}{\nu\pi}\int_0^\infty \left\{1 - exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right)\right\}\frac{\sin(y\xi)}{\xi^3}d\xi, \tag{56}$$

corresponding to the unsteady motion of a second grade fluid due to a suddenly moved plate or a constantly accelerating plate (a plate that slides in its plane with a velocity V or At). As form, these expressions are identical to those of the shear stresses $\tau_{0SG}(y,t)$ and $\tau_{1SG}(y,t)$ given by Eqs. (52) and (54) (corresponding to the motion induced by a plate that applies a shear stress f or ft to the fluid). This is not a surprise because a simple analysis of the equations (2) and (3) with $\lambda = 0$ shows that the shear stress $\tau(y,t)$ in such motions of second grade fluids satisfies the same governing equation as velocity, i.e.

$$\frac{\partial\tau(y,t)}{\partial t}=(\nu+\gamma\frac{\partial}{\partial t})\frac{\partial^2\tau(y,t)}{\partial y^2}\quad like\quad \frac{\partial v(y,t)}{\partial t}=(\nu+\gamma\frac{\partial}{\partial t})\frac{\partial^2 v(y,t)}{\partial y^2}.$$

Consequently, the present results regarding second grade fluids bring about exact solutions for the velocity $v(y,t)$ corresponding to the unsteady motion due to an infinite plate that slides in its plane with a velocity At^a.

Furthermore, eliminating $v(y,t)$ between Eqs. (2) and (3), we obtain for the shear stress $\tau(y,t)$ a governing equation

$$(1+\lambda\frac{\partial}{\partial t})\frac{\partial\tau(y,t)}{\partial t}=\nu(1+\lambda_r\frac{\partial}{\partial t})\frac{\partial^2\tau(y,t)}{\partial y^2};\quad t>0,$$

of the same form as Eq. (4) for velocity. Consequently, the present results also allow us to present close form solutions for the velocity of Maxwell and Oldroyd-B fluids over an infinite plate that is moving in its plane according to the boundary condition.

$$v(0,t)=\frac{A}{\lambda}\int_0^t (t-s)^a G_{1,0,1}(-\frac{1}{\lambda},s)ds;\quad a\geq 0.$$

Now, in order to bring to light some relevant physical aspects of the obtained results, the influence of fractional parameters on the fluid velocity is underlined by graphical illustrations. A series of calculations were performed for different solutions with typical values. The velocity of the fluid, as it results from Fig. 3 is an increasing function with respect to α. Consequently, a fractional Maxwell fluid flows slower in comparison with an ordinary Maxwell fluid. The influence of fractional parameter β on velocity is shown in Fig. 4. The velocity of the fluid is an increasing function of β in a relative small neighborhood of the plate only. Therefore, in the vicinity of the plate the fractional second grade fluid also flows slower in comparison with an ordinary second grade fluid. A comparison between Oldroyd-B and fractional Oldroyd-B models is realized in Fig. 5. As it was to be expected, for α and $\beta\to 1$ the diagrams of velocity tend to that corresponding to the Oldroyd-B fluid. The units of all material constants in Figs. 1-5 are ISI units.

Appendix A

$$L^{-1}\left\{\frac{\Gamma(a+1)}{q^a}\right\}=\left\{\begin{array}{ll}\delta(t), & a=0\\ at^{a-1}, & a>0\end{array}\right.;\quad L\left\{\frac{1}{q+\nu\xi^2}\right\}=e^{-\nu\xi^2 t},\qquad (A_1)$$

where $\delta(.)$ is the Dirac delta function and $(\delta * f)(t) = f(t)$.

$$\int_0^\infty \frac{1-cos(y\xi)}{\xi^2}d\xi = \frac{\pi}{2}y; \quad \frac{1}{z+a} = \sum_{k=0}^{\infty}(-1)^k\frac{z^k}{a^{k+1}}; \quad (b{+}1)^k = \sum_{m=0}^{k}\frac{k!b^m}{m!(k-m)!}. \qquad (A_2)$$

$$G_{a,b,c}(d,t) = L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}; Re(ac{-}b) > 0, \;\; Re(q) > 0 \;\; and \;\; |\frac{p}{q^a}| < 1. \qquad (A_3)$$

$$\lim_{\lambda\to 0}\frac{1}{\lambda^{\alpha k}}G_{a,b,k}\left(-\frac{1}{\lambda^\alpha},t\right) = \frac{t^{-b-1}}{\Gamma(-b)} \;\; if \;\; b<0. \qquad (A_4)$$

$$\frac{2}{\pi}\int_0^\infty\left[1-e^{-\nu\xi^2 t}\cos(y\xi)\right]\frac{d\xi}{\xi^2} = 2\sqrt{\frac{\nu t}{\pi}}exp\left(-\frac{y^2}{4\nu t}\right){+}yerf\left(\frac{y}{2\sqrt{\nu t}}\right). \qquad (A_5)$$

$$\int_0^t G_{a,b,c}(d,s)ds = G_{a,b-1,c}(d,t). \qquad (A_6)$$

Appendix B

$$\gamma\xi^2\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\int_0^t e^{-\nu\xi^2(t-s)}G_{1,k,k+1}(-\nu\xi^2,s)ds = \frac{1}{\nu\xi^2}\left[exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right)-e^{-\nu\xi^2 t}\right]. \qquad (B_1)$$

$$\lambda_r\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\int_0^t e^{-\nu\xi^2(t-s)}G_{1,k+1,k+1}(-\nu\xi^2,s)ds = \frac{1}{\nu\xi^2}\left[e^{-\nu\xi^2 t}-\frac{1}{1+\gamma\xi^2}exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right)\right]. \qquad (B_2)$$

$$\gamma\xi^2\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\int_0^t e^{-\nu\xi^2(t-s)}G_{1,k-1,k+1}(-\nu\xi^2,s)ds = \frac{1}{\nu^2\xi^4}\left[\gamma\xi^2+e^{-\nu\xi^2 t}-(1+\gamma\xi^2)exp\left(-\frac{\nu\xi^2 t}{1+\gamma\xi^2}\right)\right]. \qquad (B_3)$$

$$\lambda_r\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\int_0^t e^{-\nu\xi^2(t-s)}G_{1,k,k+1}(-\nu\xi^2,s)ds = \frac{1}{\nu^2\xi^4}\left[exp(-\frac{\nu\xi^2 t}{1+\gamma\xi^2})-e^{-\nu\xi^2 t}\right]. \qquad (B_4)$$

In order to prove the last two relations, we use the next identities

$$\frac{\gamma\xi^2}{q+\nu\xi^2}\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\frac{q^{k-1}}{(q+\nu\xi^2)^{k+1}} = \frac{1}{q+\nu\xi^2}\frac{\gamma\xi^2}{q[(q+\nu\xi^2)+\gamma\xi^2 q]} = \frac{1}{q+\nu\xi^2}\frac{\gamma\xi^2}{1+\gamma\xi^2}\frac{1}{q\left(q+\frac{\nu\xi^2}{1+\gamma\xi^2}\right)},$$

$$\frac{\lambda_r}{q+\nu\xi^2}\sum_{k=0}^{\infty}(-\gamma\xi^2)^k\frac{q^k}{(q+\nu\xi^2)^{k+1}}=\frac{1}{q+\nu\xi^2}\frac{\lambda_r}{q+\nu\xi^2+\gamma\xi^2 q}=\frac{1}{\nu^2\xi^4}\left[\frac{1}{q+\frac{\nu\xi^2}{1+\gamma\xi^2}}-\frac{1}{q+\nu\xi^2}\right].$$

Applying the inverse Laplace transform to the last identity, for instance, and using $(A_1)_2$ and the property $(f_1 * f_2)(t)=\int_0^t f_1(t-s)f_2(s)ds=L^{-1}\{F_1(q)F_2(q)\}$, we immediately obtain (B_4).

References

[1] T. Hayat, M. Khan, M. Ayub, Exact solutions of flow problems of an Oldroyd-B fluid, Appl. Math. Comput. 151 (2004) 105-119.

[2] W. C. Tan, T. Masuoka, Stokes' first problem for an Oldroyd-B fluid in a porous half space, Phys. Fluids 17 (2005) 023101-7.

[3] C. Fetecau, K. Kannan, A note on an unsteady flow of an Oldroyd-B fluid, Int. J. Math. Math. Sci. 19 (2005) 3185-3194.

[4] T. Hayat, M. Hussain, M. Khan, Hall effect on flows of an Oldroyd-B fluid through porous medium for cylindrical geometries, Comput. Math. Appl. 52 (2006) 269-282

[5] N. Aksel, C. Fetecau, M. Scholle, Starting solutions for some unsteady unidirectional flows of Oldroyd-B fluids, Z. Angew. Math. Phys. 57 (2006) 1-17.

[6] C. Fetecau, S. C. Prasad, K. R. Rajagopal, A note on the flow induced by a constantly accelerating plate in an Oldroyd-B fluid, Appl. Math. Model. 31 (2007) 647-654.

[7] T. Hayat, S. B. Khan, M. Khan, The influence of Hall current on the rotating flow of an Oldroyd-B fluid in a porous medium, Nonlinear Dynam. 47 (2007) 353-362.

[8] Corina Fetecau, T. Hayat, C. Fetecau, Starting solutions for Oscillating motions of Oldroyd-B fluids, J. Non-Newton. Fluid Mech. 153 (2008) 191-201.

[9] M. Hussain, T. Hayat, C. Fetecau, S. Asghar, On accelerated flows of an Oldroyd-B fluid in a porous medium, Nonlinear Anal. Real World Appl. 9 (2008) 1394-1408.

[10] S. McGinty, S. McKee, R. McDermott, Analytic solutions of Newtonian and non-Newtonian pipe flows subject to a general time-dependent pressure gradient, J. Non-Newton. Fluid. Mech. 162 (2009) 54-77.

[11] Asia Anjum, M. Ayub, M. Khan, Starting solutions for oscillating motions of an Oldroyd-B fluid over a plane wall, Commun Nonlinear Sci Number Simulat, doi: 10.1016/ j.cnsns.2011.05.004.

[12] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997.

[13] O. P. Agrawal, Solution for a fractional diffusion-wave equation defined in a bounded domain, Nonlinear Dynam. 29 (2002) 145-155.

[14] M. Seredyńska, A. Hanyga, Nonlinear differential equations with fractional damping with applications to the 1dof and 2dof pendulum, Acta Mech. 176 (2005) 169-183.

[15] D. Baleanu, S. I. Muslih, K. Tas, Fractional Hamiltonian analysis of higher order derivatives systems, J. Math. Phys. 47, 10 (2006) art. no. 103503, 8 pp.

[16] C. Friedrich, Relaxation and retardation functions of the Maxwell model with fractional derivatives, Rheol. Acta 30 (1991) 151-158.

[17] L. I. Palade, V. Verney, P. Attané, A modified fractional model to describe the entire viscoelastic behavior of polybutadienes from flow to glassy regime, Rheol. Acta 35 (1996) 265-273.

[18] A. Germant, On fractional differentials, philosophical Magazine 25 (1938) 540-549.

[19] G. L. Slonimsky, Laws of mechanical relaxation processes in polymer, J. Polym. Pol. Sym. 16 (1967) 1667-1672.

[20] R. L. Bagley, P. J. Torvik, Fractional Calculus- a differential approach to the analysis of viscoelasticity damped structures, AIAA J. 21 (1983) 742-748.

[21] R. L. Bagley, P. J. Torvik, A theoretical basis for the applications of fractional calculus of viscoelasticity, J. Rheol. 27 (1983) 201-210.

[22] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, J. Appl. Mech. 51 (1984) 299-307.

[23] N. Makris, G. F. Dargush, M. C. Constantinou, Dynamic analysis of generlized viscoelastic fluids, J. Eng. Mech. 119 (1993) 1663-1679.

[24] C. Friedrich, Relaxation functions of rheological constitutive equations with fractional derivatives: Thermodynamical constraints, in: J. Casas-Vasquez, D. Jou (Eds.), Lecture Notes in Phys., vol. 381, Springer-Verlag, New York, 1991.

[25] H. H. Kausch, N. Heymans, C. J. Plummer, P. Decroly, Matériaux Polymers: Propriétés Mécaniques et Physiques, Presses Polytechniques et Universitaires Romandes, Lausanne, 2001, p. 195.

[26] F. Mainardi, Fractional Calculus and Waves in linear viscoelasticity. An introduction to Mathematical Models, Imperial College Press, London, 2010.

[27] L. I. Palade, P. Attané, R. R. Huilgol, B. Mena, Anomalous stability behavior of a properly invariant constitutive equation which generalizes fractional derivative models, Int. J. Eng. Sci. 37 (1999) 315-329.

[28] D. Vieru, Corina Fetecau, C. Fetecau, Flow of a generalized Oldroyd-B fluid due to a costantly accelerating plate, Appl. Math. Comput. 201 (2008) 834-842.

[29] H. T. Qi, M. Y. Xu, Some unsteady unidirectional flows of a generalized Oldroyd-B fluid with fractional derivatives, Appl. Math. Model. 33 (2009) 4184-4191.

[30] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and derivatives: Theory and Applications, Gordon and Breach, Amsterdam, 1993.

[31] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[32] D. Y. Song, T. Q. Jiang, Study on the constitutive equation with fractional derivative for the viscoelastic fluids. Modified Jeffreys model and its application, Rheol. Acta 37 (1998) 512-517.

[33] C. F. Lorenzo, T. T. Hartley, Generalized Functions for Fractional Calculus, NASA/TP-1999-209424, 1999.

[34] D. Vieru, Corina Fetecau, A. Sohail, Flow due to a plate that applies an accelerated shear to a seond grade fluid between two parallel walls perpendicular to the plate, Z. Angew. Math. Phys. 62 (2011) 161-172.

[35] I. N. Sneddon, Functional Analysis, Encyclopedia of Pysics, Vol. 2, Springer, Berlin, Gottingen, Heidelberg, 1955.

[36] Nazish Shahid, Mehwish Rana, M. A. Imran, Flow of an Oldroyd-B fluid over an infinite plate subject to a time-dependent shear stress, sent for publication.

[37] I. C. Christov, C. I. Christov, Comment on "On a class of exact solutions of the equations of motion of a second grade fluid" by C. Fetecau and J. Zierep (Acta Mech. 150, 135-138, 2001), Acta Mech.

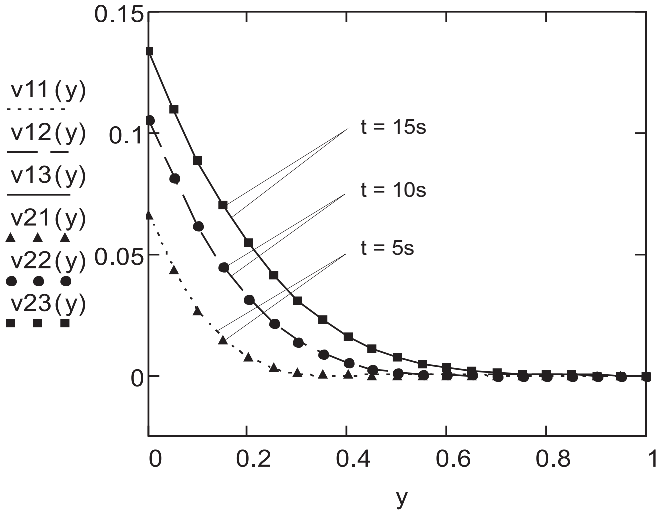

Fig. 1. Profiles of the velocity v(y,t) given by Eq. (44) - curves v11(y), v12(y), v13(y) and Eq. (45) - curves v21(y), v22(y), v23(y), for ν = 0.003974, μ = 3,902, f = -2, λ = 2, λ_r = 1 and different values of t.

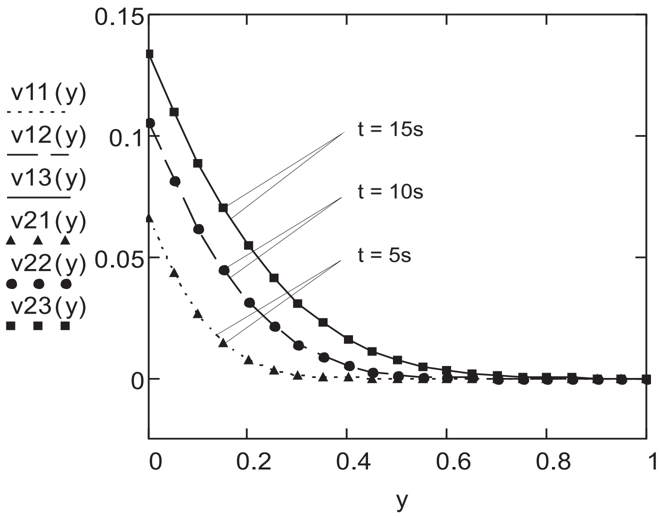

Fig. 2. Profiles of the velocity v(y,t) given by Eq. (45) - curves v11(y), v12(y), v13(y) and Eq. (46) - curves v21(y), v22(y), v23(y), for ν = 0.003974, μ = 3,902, f = -2, λ = 2, λ_r = 1 and different values of t.

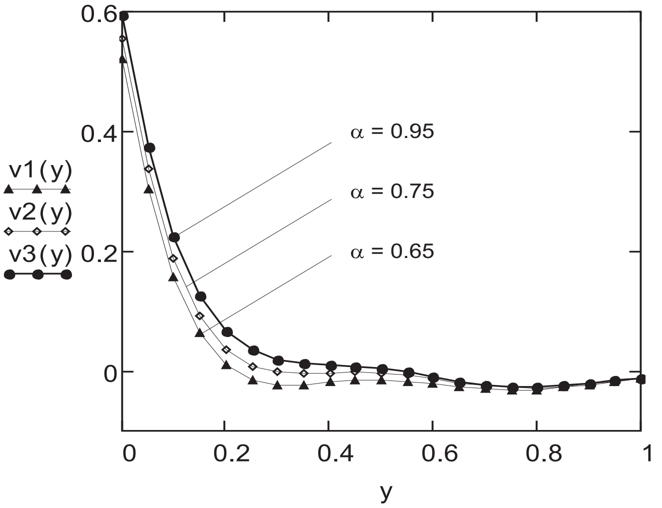

Fig. 3. Profiles of the velocity v(y,t) given by Eq. (19) - curves v1(y), v2(y), v3(y) for $a = 1$, $\nu = 0.003974$, $\mu = 3{,}902$, $f = -2$, $\lambda = 2$, $\lambda_r = 1$, $\beta = 0.2$, $t = 10s$ and different values of α.

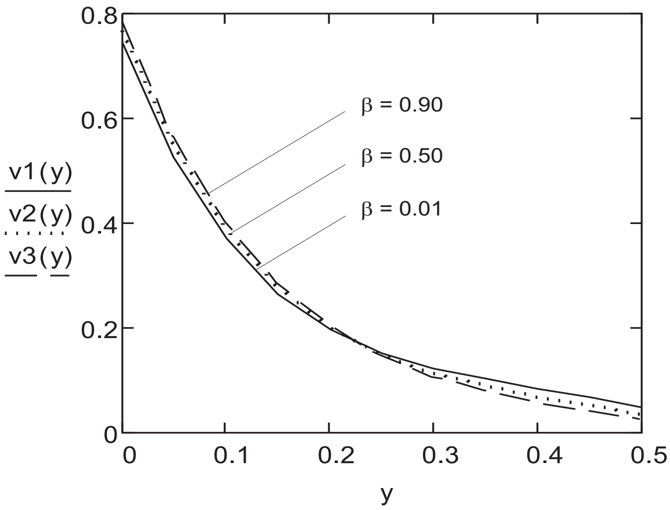

Fig. 4. Profiles of the velocity v(y,t) given by Eq. (19) - curves v1(y), v2(y), v3(y) for $a = 1$, $\nu = 0.003974$, $\mu = 3{,}902$, $f = -2$, $\lambda = 2$, $\lambda_r = 1$, $\alpha = 0.95$, $t = 10s$ and different values of β.

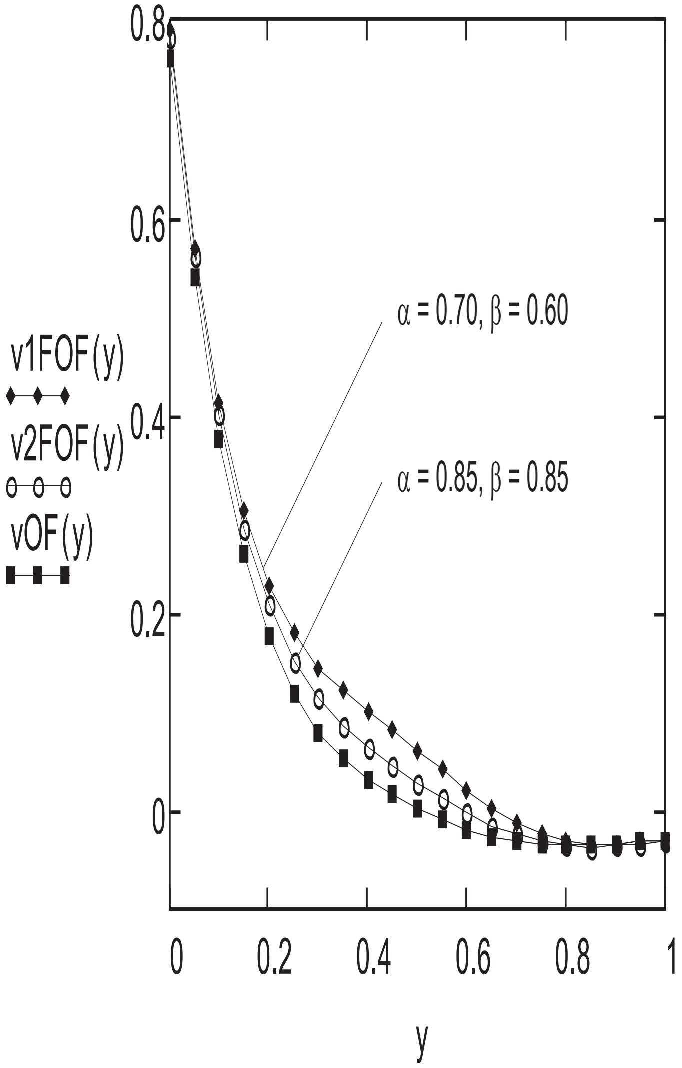

Fig. 5. Profiles of the velocity v(y,t) for fractional Oldroyd-B fluid - curves v1FOF(y), v2FOF(y) and Oldroyd-B fluid - curve vOF(y), for a = 1, with $\nu = 0.003974$, $\mu = 3{,}902$, $f = -2$, $\lambda = 2$, $\lambda_r = 1$, $t = 10s$ and different values of α and β.