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SOME COUETTE FLOWS OF A VISCOUS FLUID DUE TO TANGENTIAL STRESSES

BY

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Abstract. Couette flows of a viscous fluid produced by the motion of a wall that applies a tangential stress on the fluid are analyzed. Exact expressions for velocity are determined by means of the Laplace transform. Two particular cases, corresponding to constant and sinusoidal tangential stresses on the wall, are studied. Some relevant properties of the velocity and the volume flux are also presented.

Key words: Couette flows, viscous fluids, exact solutions.

1. Introduction

The motion of a fluid can be obtained as a result of several effects such as various types of motion of the boundaries, applications of a body force, wall that applies a tangential stress on the fluid or application of a pressure gradient. One or two of these effects can be applied together to the fluid. Unsteady flow of a viscous fluid over a flat plate in absence of side walls have been investigated by many authors, (Erdogan, 1997), (Zeng & Weinbaum, 1995), (Erdogan, 2000), (Fetecau *et al*. 2008), while the effects of side walls on steady and unsteady flows have been studied in (Sharman, 1990), (Erdogan, 1998),

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(Erdogan & Imrak, 2005). The flow of a fluid is termed as Couette flow if the fluid is bounded by two parallel walls that are in the relative motion. The flow between two parallel plates, one of which being at rest and the other one moving in its own plane with a constant speed, is called the simple Couette flow. The flow between two parallel plates produced by a constant pressure gradient in the direction of the flow is called Poiseuille flow. The general case of the Couette flow, or the generalized Couette flow, is a superposition of the simple Couette flow over the Poiseuille flow (Schlichting, 1968). Some practical applications of this type of flows have been presented in the reference (Erdogan, 1998). The unsteady Couette flow problem has been considered in several works containing various effects. Jha, (Jha, 2001) introduced both magnetic and natural convection effects on the Couette flow between two vertical plates. The effects of fluid slippage at the boundary for Couette flow are considered in the paper of (Marques *et al.*, 2000) under steady state conditions and only for gases. Khaled and Vafai, (Khaled & Vafai, 2004), have studied the effect of slip condition on Couette flows due to an oscillating wall. Other interesting results regarding flows of Newtonian or non-Newtonian fluids can be found in the references (Vieru & Rauf, 2011), (Hayat *et al.*, 2007), (Sastry *et al.*, 2010), (Jordan & Puri, 2002), (Jordan, 2005), (Christov, 2010), (Kai-Long Hsiao, 2011). This paper deals with the Couette flows of a Newtonian fluid caused by the bottom plate which applies a tangential stress $\tau_{w}(t)=\tau_0 f(t)$ on the fluid. Exact expressions for velocity are determined by means of a Laplace transform. Two particular cases, namely constant tension on the bottom plate and sinusoidal oscillations of the wall tension, are studied. Some relevant properties of the velocity are presented using graphical illustrations generated by the software Mathcad.

2. Problem Formulation and Solution

Consider an incompressible, homogeneous Newtonian fluid fill the slab $y\in(0,h)$ between two flat, infinite solid plates which are situated in the planes y = 0, and $y = h$ of a Cartesian coordinate system *Oxyz* with the positive y-axis in the upward direction, Fig.1.

Initially, both the fluid and the plate are at rest. At the moment $t = 0^+$ the fluid is set in motion by the bottom plate that applies a tangential stress $\tau_w(t)=\tau(y,t)\big|_{y=0}=\tau_0 f(t)$ to the fluid. Here, $f(t)$ is a piecewise continuous function defined on $[0,\infty)$ and $f(0)=0$. Also, we suppose that the Laplace transform of function $f(t)$ exists.

The velocity vector corresponding to such a motion has the form $\mathbf{v}=(u(y,t),0,0))$ and the constitutive and governing equations are (Erdogan, 1998; Marques *et al.*, 2000)

$$\tau(y,t)=\mu\frac{\partial u(y,t)}{\partial y}, \tag{1}$$

$$\frac{\partial u(y,t)}{\partial t}=\nu\frac{\partial^2 u(y,t)}{\partial y^2}, \qquad (y,t)\in(0,h)\times(0,\infty), \tag{2}$$

where $\tau(y,t)$ is the tangential shear stress, μ is the dynamic viscosity of the fluid, $\nu=\mu/\rho$ is the kinematic viscosity, ρ being the constant density of the fluid. The boundary and initial conditions are

$$\tau(0,t)=\mu\frac{\partial u(0,t)}{\partial y}=\tau_0 f(t), \qquad t\geq 0, \tag{3}$$

$$u(h,t)=0, \quad t\geq 0, \tag{4}$$

$$u(y,0)=0,\ y\in[0,h]. \tag{5}$$

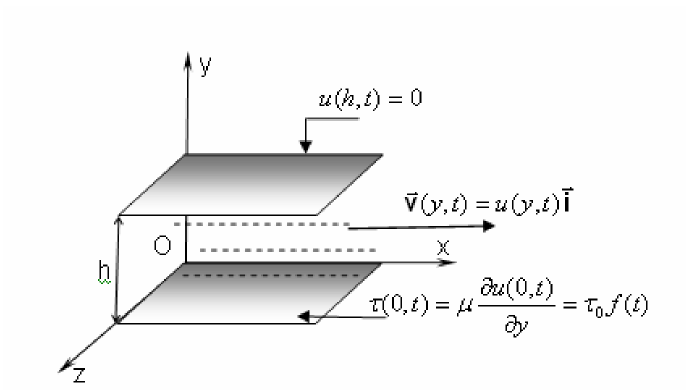

Fig. 1 − Geometry flow.

By using the following dimensionless variables and functions

$$t^*=\frac{\nu t}{h^2},\quad y^*=\frac{y}{h},\quad \tau^*=\frac{\tau}{\tau_0},\quad u^*=\frac{u}{(h\tau_0/\mu)},\quad g(t^*)=f\left(\frac{h^2t^*}{\nu}\right), \tag{6}$$

we obtain the non dimensionalized initial-boundary value problem (dropping the "*" notation)

$$\tau(y,t)=\frac{\partial u(y,t)}{\partial y}, \tag{7}$$

$$\frac{\partial u(y,t)}{\partial t}=\frac{\partial^2 u(y,t)}{\partial y^2}, \quad (y,t)\in(0,1)\times(0,\infty), \tag{8}$$

$$\tau(0,t)=\frac{\partial u(0,t)}{\partial y}=g(t), \quad t\geq 0, \tag{9}$$

$$u(1,t)=0, \quad t\geq 0, \tag{10}$$

$$u(y,0)=0, \quad y\in[0,1]. \tag{11}$$

By applying the temporal Laplace transform L, (Carslaw & Jaeger, 1963)) to Eqs. (8)-(10 and employing the initial condition (11) we obtain the problem

$$\frac{\partial^2 \overline{u}(y,q)}{\partial y^2}=q\overline{u}(y,q), \tag{12}$$

$$\frac{\partial u(0,q)}{\partial y}=G(q), \quad \overline{u}(1,q)=0, \tag{13}$$

where $\overline{u}(y,q)=L\{u(y,t)\}$, $G(q)=L\{g(t)\}$ are the Laplace transforms of functions $u(y, t)$ and $g(t)$, respectively.

The transform domain solution of Eq. (12) with the boundary conditions (13) is given by

$$\overline{u}(y,q)=G(q)G_1(y,q), \tag{14}$$

where

$$G_1(y,q)=\frac{\text{sh}[(y-1)\sqrt{q}]}{\sqrt{q}\,\text{ch}(\sqrt{q})}. \tag{15}$$

The singular points of the function $G_1(y,t)$ are simple poles located at

$$q_n = -\alpha_n^2, \quad \alpha_n = \frac{(2n+1)\pi}{2}, \quad n = 0,1,2,... \tag{16}$$

Inverting the function $G_1(y,t)$ by using the residue theorem to evaluate the Laplace inversion integral (Khaled & Vafai, 2004), we obtain

$$g_1(y,t) = L^{-1}\{G_1(y,q)\} = \sum_{n=0}^{\infty} \Re es\left[G_1(y,q)e^{qt}; q_n\right] = -2\sum_{n=0}^{\infty} \cos(\alpha_n y)\exp(-\alpha_n^2 t) \tag{17}$$

2.1. Constant Tension on the Bottom Plate

In this section we consider *g*(*t*) = *H*(*t*), where *H*(*t*) is the Heaviside step unit function. Using Eqs. (17), (14) and the convolution theorem we obtain the exact (*y*,*t*)-domain solution of the set of Eqs. (8)-(11) given by

$$u(y,t) = (g * g_1)(t) = \int_0^t g(t-s)g_1(y,s)\mathrm{d}s \tag{18}$$

and, using (A_1) from the Appendix A, we obtain the velocity field

$$u(y,t) = H(t)\int_0^t g_1(y,s)\mathrm{d}s = H(t)\left[(y-1) + 2\sum_{n=0}^{\infty} \frac{1}{\alpha_n^2}\cos(\alpha_n y)\exp(-\alpha_n^2 t)\right]. \tag{19}$$

The velocity given by Eq. (19) has the following temporal limits:

$$\lim_{t\to 0^+} u(y,t) = 0, \quad \lim_{t\to\infty} u(y,t) = y-1. \tag{20}$$

The same limits can also be obtained using the known relations $\lim\limits_{t\to 0+} u(y,t) = \lim\limits_{q\to\infty} q\overline{u}(y,q)$ and $\lim\limits_{t\to\infty} u(y,t) = \lim\limits_{q\to 0} q\overline{u}(y,q)$.

As a result from Eqs. (20), we have that the velocity *u*(*y,t*) does not exhibit a jump of discontinuity at t =0 and, for $t \to \infty$ it reduces to the "stationary solution" $u_s = y-1$. In Fig. 2 we plotted the velocity *u*(*y,t*) given by Eq. (19), versus t for $y \in \{0, 0.3, 0.7, 0.9\}$ and versus y for

$t \in \{0.05, 0.1, 0.2, 0.5\}$. It is clear that for a given value of *y*, the velocity *u*(*y,t*) decreases as function of *t* and tends to the "stationary velocity" $u_s = y - 1$ for increasing *t*. For a given value of *t*, the velocity is an increasing function with respect to *y*.

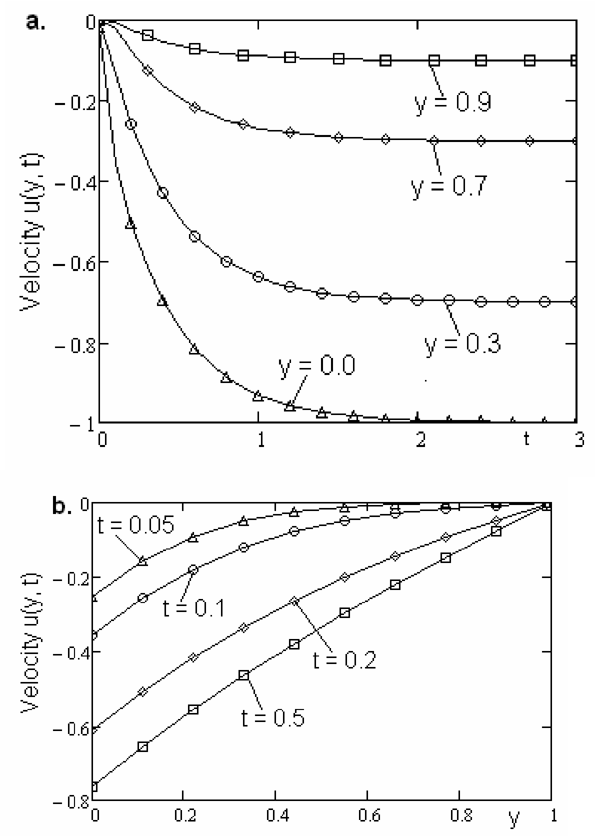

Fig. 2 − Plot of *u*(*y, t*) versus *t* (*a*) and versus *y* (*b*) for constant tension on the wall.

In addition, let us give another expression for the velocity *u*(*y*,*t*). For this, we rewrite Eq. (15) in terms of exponentials as

$$G_1(y,q) = \frac{1}{\sqrt{q}}\left[\mathrm{e}^{-(2-y)\sqrt{q}} - \mathrm{e}^{-y\sqrt{q}}\right](1+\mathrm{e}^{-2\sqrt{q}})^{-1} = \\ = \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{q}}\left[\mathrm{e}^{-(2-y+2k)\sqrt{q}} - \mathrm{e}^{-(y+2k)\sqrt{q}}\right]. \tag{21}$$

Inverting term by term Eq. (21) and using (A_2) from the Appendix A, we have

$$g_1(y,t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{\pi t}}\left[\exp\left(-\frac{(2-y+2k)^2}{4t}\right)-\exp\left(-\frac{(y+2k)^2}{4t}\right)\right]. \tag{22}$$

Now, using Eqs. (18) and (22), we obtain a new expression for the velocity $u(y,t)$, namely

$$u(y,t)=\sum_{k=0}^{\infty}(-1)^{\mathrm{k}}\left[2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{(2-y+2k)^2}{4t}\right)-(2-y+2k)\operatorname{erfc}\left(\frac{2-y+2k}{2\sqrt{t}}\right)-\right.$$
$$\left.-2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{(y+2k)^2}{4t}\right)+(y+2k)\operatorname{erfc}\left(\frac{y+2k}{2\sqrt{t}}\right)\right]H(t), \tag{23}$$

where erfc(·) is the complementary error function of Gauss. Obviously, as it result from Fig. 3, the two solutions given by Eqs. (19) and (23) are equivalent.

Finally, we determine the volume flux given by

$$Q(t)=\int_0^1 u(y,t)\mathrm{d}y=H(t)\left[-\frac{1}{2}+2\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha_n^3}\exp(-\alpha_n^2 t)\right]. \tag{24}$$

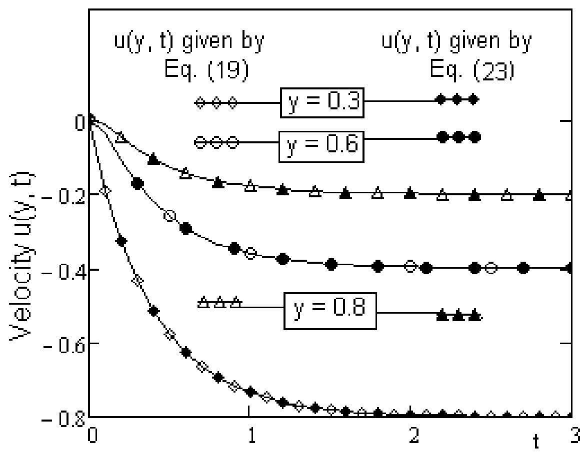

Fig. 3 − Plot of $u(y, t)$ given by Eqs. (19) and (23), for constant tension on the wall.

2.2. Sinusoidal Tension on the Bottom Plate

Let us consider $f(t)=\sin(\omega t), \omega>0$ being the frequency of oscillations of the tension on the lower plate. Using Eq. (6) we obtain, after dropping the "*" notation,

$$g(t)=\sin(\Omega t), \qquad \Omega=\frac{h^2\omega}{v}. \tag{25}$$

In this case, Eq. (14) becomes

$$\bar{u}(y,q)=\frac{\Omega}{q^2+\Omega^2}G_1(y,q).$$

The poles of function $\bar{u}(y,q)$ are $\pm\Omega$ and q_n given by (16).

Using the residue theorem, after lengthy but straightforward computations, the exact (y,t)-domain solution is

$$u(y,t)=L^{-1}\left\{\frac{\Omega}{q^2+\Omega^2}G_1(y,q)\right\}=\frac{A_1(y)\cos(\Omega t)+\mathrm{A}_2(y)\sin(\Omega t)}{\sqrt{2\Omega}\left[\cos^2\left(\frac{\Omega}{2}\right)+\mathrm{sh}^2\left(\frac{\Omega}{2}\right)\right]}- \\ -2\Omega\sum_{n=0}^{¥}\frac{\cos(\alpha_n y)}{\alpha_n^4+\Omega^2}\exp(-\alpha_n^2 t), \tag{26}$$

where

$$\begin{aligned}A_{1,2}(y)=\ &\mathrm{ch}\Omega_1\cos\Omega_1\{\mathrm{ch}[\Omega_1(y-1)]\sin[\Omega_1(y-1)]+\\ &+m\,\mathrm{sh}[\Omega_1(\mathrm{y}-1)]\cos[\Omega_1(\mathrm{y}-1)]\}-\\ &-\mathrm{sh}\Omega_1\sin\Omega_1\{\mathrm{sh}[\Omega_1(y-1)]\cos[\Omega_1(y-1)]\pm\\ &\pm\mathrm{ch}[\Omega_1(y-1)]\sin[\Omega_1(y-1)]\},\end{aligned} \tag{27}$$

and $\Omega_1=\sqrt{\frac{\Omega}{2}}$.

The temporal limits of the velocity given by Eq. (26) are

$$\lim_{t\to 0^+} u(y,t)=0, \quad \lim_{t\to\infty} u(y,t)=\frac{A_1(y)\cos(\Omega t)+A_2(y)\sin(\Omega t)}{\sqrt{2\Omega}\left[\cos^2\left(\frac{\Omega}{2}\right)+\text{sh}^2\left(\frac{\Omega}{2}\right)\right]}=u_p(y,t)\cdot \tag{28}$$

For large values of time *t,* the velocity u(y,t) given by Eq. (26) reduces to the "permanent solution" given by Eq. $(28)_2$.

Integrating Eq. (26) with respect to $y\in[0,1]$, we obtain the volume flux

$$Q(t)=\frac{1}{\Omega}-\frac{\text{ch}(\Omega_1)\cos(\Omega_1)\cos(\Omega t)+sh(\Omega_1)\sin(\Omega_1)\sin(\Omega t)}{\Omega\left[\cos^2(\Omega_1)+sh^2(\Omega_1)\right]}-$$
$$-\frac{64\Omega}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\left[(2n+1)^4\pi^4+(4\Omega)^2\right]}\exp\left[-\frac{(2n+1)^2\pi^2}{4}t\right], \tag{29}$$

which, for large values of time *t*, reduces to the "permanent flux"

$$Q_p(t)=\frac{1}{\Omega}-\frac{\text{ch}(\Omega_1)\cos(\Omega_1)\cos(\Omega t)+sh(\Omega_1)\sin(\Omega_1)\sin(\Omega t)}{\Omega\left[\cos^2(\Omega_1)+sh^2(\Omega_1)\right]}. \tag{30}$$

Some properties of the flow are revealed in Fig. 4. This figure contains diagrams of velocity $u(y,t)$ given by Eq. (26) and for the permanent velocity $u_p(y,t)$ given by Eq. $(28)_2$, versus *t* for $y\in\{0.7, 0.9\}$, respectively, the diagrams of velocity $u(y,t)$ versus y and $t\in\{0.3, 0.6, 1.0, 1.2\}$.

From these figures, it is evident that, the difference between the velocity $u(y,t)$ and the permanent velocity given by Eq. $(28)_2$ is significant only for small values of the time *t.* We see that, in the considered case $\Omega=1.5$, after the moment t=2, the transient velocity $u_t(y,t)=u(y,t)-u_p(y,t)$ can be neglected. For small values of the time *t*, as it result from Fig.4*b*, the velocity $u(y,t)$ is an increasing function with regards to *y*.

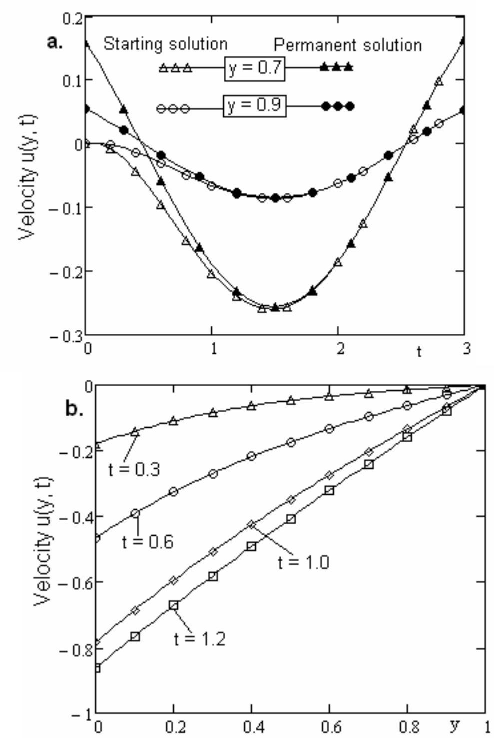

Fig. 4 − Plot of $u(y, t)$ versus t (Fig. 4*a*) and versus y (Fig. 4*b*) for sinusoidal tension on the wall.

3. Conclusions

1. Couette flows of Newtonian fluids have been analyzed in the assumption that the bottom plate, situated in the plane $y = 0$, applies a tangential stress to the fluid. Two particular cases, corresponding to constant and sinusoidal shear stresses on the wall, were considered. Exact expressions for the velocity $u(y,t)$ have been determined by means of the Laplace transform. Some properties of the velocity $u(y,t)$ were presented. In the case of a constant tangential tension on the bottom plate, the velocity $u(y,t)$ is an increasing function on y. For large values of the time t the velocity tends to the"stationary velocity" $u_s = y-1$ (Fig. 2).

2. For this case, two different equivalent expressions for velocity have been established. This equivalence is shown by Fig. 3. If the plate applies a sinusoidal shear stress on the fluid, the velocity $u(y,t)$ is written as a sum between the"the permanent solution" $u_p(y,t)$ and the transient solution $u_t(y,t) = u(y,t) - u_p(y,t)$. For large values of the time t, the transient velocity

can be neglected and the fluid flows according to the"permanent solution" $u_p(y,t)$, (Fig. 4). In both cases the volume flux was determined.

Appendix A

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\left[\frac{(2n+1)\pi}{2}y\right]=\frac{\pi^2}{8}(1-y),\qquad y\in[0,1]\,. \tag{A$_1$}$$

$$L^{-1}\left\{\frac{e^{-a\sqrt{q}}}{\sqrt{q}}\right\}=\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{a^2}{4t}\right),\quad \Re e(a^2)\geq 0 \tag{A$_2$}$$

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CURGERI COUETTE ALE UNUI FLUID VÂSCOS PRODUSE DE TENSIUNI TANGENȚIALE

(Rezumat)

Se studiază curgerile Couette ale unui fluid vâscos produse de mişcarea unui perete ce aplică fluidului tensiuni tangențiale. Se obțin soluții exacte pentru viteză utilizând transformata Laplace. Se analizează două cazuri particulare ce corespund la tensiuni tangențiale aplicate de tip constant şi sinusoidale.