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FLOW OF AN OLDROYD-B FLUID OVER AN INFINITE PLATE SUBJECT TO A TIME-DEPENDENT SHEAR STRESS

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ABSTRACT. The velocity field and the shear stress corresponding to the unsteady flow of an Oldroyd-B fluid due to an infinite flat plate, subject to a time-dependent shear stress, are established in integral form using the Fourier cosine transform. Similar solutions for Maxwell, Second grade and Newtonian fluids are recovered as limiting cases of general solutions. These solutions satisfy both the governing equations and all imposed initial and boundary conditions. Finally, a comparison between the four models as well as the influence of the pertinent parameters on the fluid motion is underlined by graphical illustrations.

Key words: unsteady motion, Oldroyd-B fluids, time-dependent shear stress.
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1. Introduction

The study of the motion of a fluid over an infinite plate is of interest both for academic research and due to its practical importance. Stokes [1] solved the problem for a viscous fluid flow and Soundalgekar [2] extended it to a fluid of second grade using a perturbation method. Teipel [3] showed that for such a fluid, a similarity solution does not exist and provided a series solution. Puri [4] studied the problem using the Laplace transform and found a solution which does not satisfy the initial condition. Later, Bandelli et al [5] and Bandelli and Rajagopal [6] showed that the soluations obtained using the Laplace transform do not satisfy the intial conditions. Moreover, the corresponding

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Navier- Stokes solutions cannot be retrieved from those for fluids of second grade. In the last time, there have been several papers devoted to the study of such flows of non-Newtonian fluids [7-21]. However, it is worth pointing out that the most of them dealt with problems for which the velocity is given on the boundary. To the best of our knowledge, the first exact solutions for such flows of non-Newtonian fluids in which the shear stress is given on the plate are those obtained by Bandelli et al [5]. The first exact solutions for the motion of a non-Newtonian fluid due to an infinite circular cylinder subject to a constant longitudinal/rotational shear stress are those of Bandelli and Rajagopal [6]. Other exact solutions for motions of second grade or Oldroyd-B fluids, induced by an infinite plate subject to a shear stress, have been obtained by Erdogan [10], Fetecau and Kannan [13] and Vieru et al [21].

The aim of this note is to provide exact solutions for the velocity field and the shear stress corresponding to the unsteady motion of an Oldroyd-B fluid past an infinite flat plate subject to a time-dependent shear stress. These solutions, obtained by means of the Fourier cosine transform and presented in terms of some definite integrals, satisfy the governing equations and all imposed initial and boundary conditions. They can be easy specialized to give the similar solutions for Maxwell, Second grade and Newtonian fluids performing the same motion. Since all solutions obtained here are mathematically exact, it is expected that they may offer help for further analytical and experimental research on non-Newtonian fluids. Such solutions can be also used as tests to verify numerical schemes that are developed to study more complex unsteady flow problems. Just as in the case of Newtonian fluids, it is necessary to develop a large class of exact and approximate solutions for Oldroyd-B fluids as they have been found to approximate the response of many dilute polymeric liquids. However, it is worth pointing out that the equations of motion of Oldroyd-B fluids are in general of higher order than the Navier-Stokes equations. Thus, in order to obtain exact solutions to these equations, in general, we must require boundary/initial conditions in addition to the usual no slip conditions. Further the nonlinearities which occur in the equations of motion of such fluids are also of higher order. In this context we refer the reader to [22, 23] for further discussion.

Finally, some characteristics of the behavior of Newtonian, Second grade, Maxwell and Oldroyd-B fluids as well as the influence of pertinent parameters on the fluid motion are underlined by graphical illustrations.

2. Formulation of the Problem and Governing Equations

The Cauchy stress $\mathbf{T}$ corresponding to an incompressible Oldroyd-B fluid is given by the constitutive equations [13, 15, 18, 19, 20, 24]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T)], \quad (1)$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, μ is the dynamic viscosity, λ and λ_r are relaxation and retardation times, the superscript T indicates the transpose operation and the superposed dot denotes the material time derivative. The model characterized by the constitutive equations (1) contains as special cases the upper-convected Maxwell model for $\lambda_r = 0$ and the Newtonian fluid model for $\lambda = \lambda_r = 0$. In some special cases, like that to be here considered, the governing equations for Oldroyd fluids reseamble those for second grade fluids. Consequently, for such flows, the solutions for second grade fluids as well as those for Maxwell and Newtonian fluids can be obtained as limiting cases of general solutions corresponding to Oldroyd-B fluids. For the problem under consideration we assume a velocity field $\mathbf{v}$ and an extrastress $\mathbf{S}$ of the form [11, 13, 18]

$$\mathbf{v} = \mathbf{v}(y,t) = v(y,t)\mathbf{i}, \ \mathbf{S} = \mathbf{S}(y,t), \tag{2}$$

where $\mathbf{i}$ is the unit vector in the x - direction of the Cartesian coordinate system x, y and z. For such flows the constraint of incompressibility is automatically satisfied while the governing equations, in the absence of body forces and a pressure gradient in the flow direction, are given by [11, 13, 18]

$$\lambda\frac{\partial^2 v(y,t)}{\partial t^2} + \frac{\partial v(y,t)}{\partial t} = \nu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2 v(y,t)}{\partial y^2}, \tag{3}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(y,t) = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\frac{\partial v(y,t)}{\partial y}, \tag{4}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, ρ is the constant density of the fluid and $\tau(y,t) = S_{xy}(y,t)$ is the non-trivial shear stress.

The partial differential equations (3) and (4), with adequate initial and boundary conditions, can be solved in principle by different methods, their effectiveness strictly depending of the domain of definition. In our case, the integral transforms technique represents a systematic, efficient and powerful tool. The Fourier cosine transform, for instance, can be used to eliminate the spatial variable into Eq. (3).

Let us now consider an incompressible Oldroyd-B fluid at rest over an infinitely extended plate which is situated in the plane $y = 0$. After the moment $t = 0^+$, the plate is suddenly pulled with a time-dependent shear in its plane

$$\tau(0,t) = f\left[t - \lambda\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right]\right]; \ \ t \geq 0, \tag{5}$$

where f is a negative constant [10]. Due to the shear the fluid above the plate is gradually moved its velocity being of the form $(2)_1$. The governing equations are given by Eqs. (3) and (4) while the appropriate initial and boundary conditions are

$$v(y,0) = \frac{\partial v(y,0)}{\partial t} = 0;\ \tau(y,0) = 0 \text{ for } y > 0, \tag{6}$$

$$\Big[\tau(y,t) + \lambda\frac{\partial \tau(y,t)}{\partial t}\Big]_{y=0} = \Big[\mu\Big(1+\lambda_r\frac{\partial}{\partial t}\Big)\frac{\partial v(y,t)}{\partial t}\Big]_{y=0} = ft;\quad t \geq 0, \tag{7}$$

Furthermore, the natural condition at infinity [22, 23]

$$v(y,t) \to 0 \text{ for } y \to \infty \text{ and } t > 0, \tag{8}$$

has to be also satisfied. Of course, the boundary condition (5) is just the solution of the ordinary differential equation (7) in $\tau(y,t)$. For $\lambda \to 0$, Eq. (5) reduces to the simple form

$$\tau(0,t) = ft;\quad t \geq 0, \tag{9}$$

corresponding to Newtonian and second grade fluids. Special case corresponding to f instead of ft in Eqs. (7) or (9), has been considered by Bandelli et al [5], Erdogan [10] for second grade fluids and Fetecau and Kannan [13] for Oldroyd-B fluids.

3. Exact Solutions

3.1. **Calculation of the velocity field.** Multiplying both sides of Eq. (3) by $\sqrt{\frac{2}{\pi}}\cos(y\xi)$, integrating with respect to y from 0 to ∞ and taking into account the initial and boundary conditions (6)-(8), we find that

$$\lambda\frac{\partial^2 v_c(\xi,t)}{\partial t^2} + (1+\alpha\xi^2)\frac{\partial v_c(\xi,t)}{\partial t} + \nu\xi^2 v_c(\xi,t) = -\sqrt{\frac{2}{\pi}}\frac{f}{\rho}t;\quad \xi, t > 0, \tag{10}$$

where $\alpha = \nu\lambda_r$ and the Fourier cosine transform $v_c(\xi,t)$ of $v(y,t)$ has to satisfy the initial conditions

$$v_c(\xi,0) = \frac{\partial v_c(\xi,0)}{\partial t} = 0;\ \ \xi > 0. \tag{11}$$

When ξ is regarded as a constant parameter, Eq. (10) may be treated as an ordinary differential equation. Its solution subject to the initial conditions (11) is

$$v_c(\xi,t) = -\sqrt{\frac{2}{\pi}}\frac{f}{\mu}\frac{1}{\xi^2}\Bigg[t - \frac{1+\alpha\xi^2}{\nu\xi^2}\Bigg(1 - \frac{r_2\exp(r_1 t) - r_1\exp(r_2 t)}{r_2 - r_1}\Bigg) - \tag{12}$$

$$- \frac{\exp(r_2 t) - \exp(r_1 t)}{r_2 - r_1}\Bigg],$$

where $r_1, r_2 = \frac{-(1+\alpha\xi^2)\pm\sqrt{(1+\alpha\xi^2)^2-4\nu\lambda\xi^2}}{2\lambda}$.
Inverting Eq. (12) by means of Fourier's cosine formula [26] and using the known
result $\int_0^\infty \frac{\sin^2 y}{y^2}dy = \frac{\pi}{2}$, we find the velocity field $v(y,t)$ in the form
$v(y,t) =$

$$\frac{fy}{\mu}(t-\lambda_r) - \frac{2f}{\mu\pi}\int_0^\infty \Bigg[(t-\lambda_r) - \Bigg(1-(1+\alpha\xi^2)\frac{r_2\exp(r_1t)-r_1\exp(r_2t)}{r_2-r_1}+ \tag{13}$$

$$+\nu\xi^2\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}\Bigg)\frac{\cos(y\xi)}{\nu\xi^2}\Bigg]\frac{1}{\xi^2}d\xi.$$

Of course, Eq. (13) can further be processed to give the simpler form
$v(y,t) =$

$$\frac{fy}{\mu}(t-\lambda_r)-\frac{2f}{\mu\pi}\int_0^\infty\Bigg[(t-\lambda_r)-\Bigg(1-\lambda\frac{(r_1^2\exp(r_2t)-r_2^2\exp(r_1t))}{r_2-r_1}\Bigg)\frac{\cos(y\xi)}{\nu\xi^2}\Bigg]\frac{1}{\xi^2}d\xi. \tag{14}$$

3.2. **Calculation of the shear stress.** Solving Eq. (4) with respect to $\tau(y,t)$ and having in mind the initial condition $(6)_3$, we find that

$$\tau(y,t) = \frac{\mu}{\lambda}\exp\left(-\frac{t}{\lambda}\right)\int_0^t \exp\left(\frac{\tau}{\lambda}\right)\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial v(y,\tau)}{\partial y}d\tau. \tag{15}$$

Substituting Eq. (14) into Eq. (15) we obtain after lengthy but straightforward computations, the next simple form $\tau(y,t) =$

$$f\left[t-\lambda\left(1-\exp\left(-\frac{t}{\lambda}\right)\right)\right]-\frac{2f}{\nu\pi}\int_0^\infty\left[1-\frac{r_2\exp(r_1t)-r_1\exp(r_2t)}{r_2-r_1}\right]\frac{\sin(y\xi)}{\xi^3}d\xi \tag{16}$$

for the shear stress.

4. Limiting Cases

1. Making $\lambda_r \to 0$ into Eqs. (14) and (16), we obtain the solutions (cf. [27, Eqs. (25) and (28)])

$$v_M(y,t) = \frac{fy}{\mu}t - \frac{2f}{\mu\pi}\int_0^\infty\Big[(t-\Big(1-\lambda\frac{r_3^2\exp(r_4t)-r_4^2\exp(r_3t)}{r_4-r_3}\Big)\frac{\cos(y\xi)}{\nu\xi^2}\Big]\frac{1}{\xi^2}d\xi, \tag{17}$$

$$\tau_M(y,t) =$$
$$f\left[t-\lambda\left(1-\exp\left(-\frac{t}{\lambda}\right)\right)\right]-\frac{2f}{\nu\pi}\int_0^\infty\left[1-\frac{r_4\exp(r_3t)-r_3\exp(r_4t)}{r_4-r_3}\right]\frac{\sin(y\xi)}{\xi^3}d\xi, \tag{18}$$

corresponding to a Maxwell fluid performing the same motion. Into above relations

$$r_3, r_4 = \frac{-1\pm\sqrt{1-4\nu\lambda\xi^2}}{2\lambda}.$$

2. By now letting $\lambda \to 0$ into Eqs. (14) and (16), the solutions (cf. [27, Eqs. (13) and (14)])

$$v_{SG}(y,t) =$$
$$\frac{fy}{\mu}(t-\lambda_r)-\frac{2f}{\mu\pi}\int_0^\infty\left[(t-\lambda_r)-\left(1-(1+\alpha\xi^2)\exp(-\frac{\nu\xi^2t}{1+\alpha\xi^2})\right)\frac{\cos(y\xi)}{\nu\xi^2}\right]\frac{1}{\xi^2}d\xi, \tag{19}$$

$$\tau_{SG}(y,t) = ft-\frac{2f}{\nu\pi}\int_0^\infty\left[1-\exp(-\frac{\nu\xi^2t}{1+\alpha\xi^2})\right]\frac{\sin(y\xi)}{\xi^3}d\xi, \tag{20}$$

corresponding to second grade fluids are recovered.

3. Finally, making $\lambda \to 0$ into Eqs. (17) and (18) or $\alpha \to 0$ into Eqs. (19) and (20), the solutions

$$v_N(y,t) = \frac{fy}{\mu}t-\frac{2f}{\mu\pi}\int_0^\infty\left[t-\left(1-\exp(-\nu\xi^2t)\right)\frac{\cos(y\xi)}{\nu\xi^2}\right]\frac{1}{\xi^2}d\xi, \tag{21}$$

$$\tau_N(y,t) = ft-\frac{2f}{\nu\pi}\int_0^\infty\left[1-\exp(-\nu\xi^2t)\right]\frac{\sin(y\xi)}{\xi^3}d\xi, \tag{22}$$

corresponding to a Newtonian fluid are recovered. Eq.(22), as it was shown into [27], is equivalent to the simple form

$$\tau_N(y,t) = f\int_0^t erfc\left(\frac{y}{2\sqrt{\nu s}}\right)ds, \tag{23}$$

where erfc(.) is the complementary error function of Gauss. Of course, the last solutions given by Eqs. (19)-(23) correspond to the boundary condition (9).

5. Numerical Results and Conclusions

In this note, the velocity field and the adequate shear stress corresponding to the unsteady motion of an Oldroyd-B fluid over an infinite flat plate subject to a time-dependent shear stress are determined using the Fourier cosine transform. Direct computations show that $v(y,t)$ and $\tau(y,t)$, given by Eqs. (14) and (16), satisfy both the governing equations and all imposed initial and boundary conditions. Furthermore, for $\lambda \to 0$, $\lambda_r \to 0$ or both λ_r and

$\lambda \to 0$ the solutions that have been obtained reduce to the similar solutions corresponding to Maxwell, Second grade or Newtonian fluids, respectively.

Finally, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity field $v(y,t)$ and the shear stress $\tau(y,t)$ are depicted against y for different values of t and of the material constants. For comparison, in Fig. 1 are presented the diagrams of the velocity and the shear stress corresponding to the four models at two different times. For velocity, roughly speaking, there are two distinct intervals in which the behavior of the four fluids is completely different. Near the plate the Newtonian fluid is the swiftest while the fluid of Oldroyd type is the slowest. In the second part of the flow domain, as usual, the second grade fluid is the swiftest and the Maxwell one is the slowest. As regards the shear stress, into absolute value, it is the biggest for second grade fluids and smallest for Maxwell fluids on the whole domain. Figs. 2, 3 and 4 show the effect of the kinematic viscosity ν and of the relaxation and retardation times λ and λ_r on the velocity and the shear stress of the Oldroyd-B fluid. Near the plate, as it result from Figs. 2a and 4a, the velocity of the fluid decreases for increasing ν or λ_r . On the other part of the flow domain the velocity is an increasing function with resppect to ν and λ_r . The shear stress is an increasing function with regard to ν and λ_r on the whole domain. Figs. 3a and 3b, clearly show, that both the velocity and the shear stress are decreasing functions of λ .

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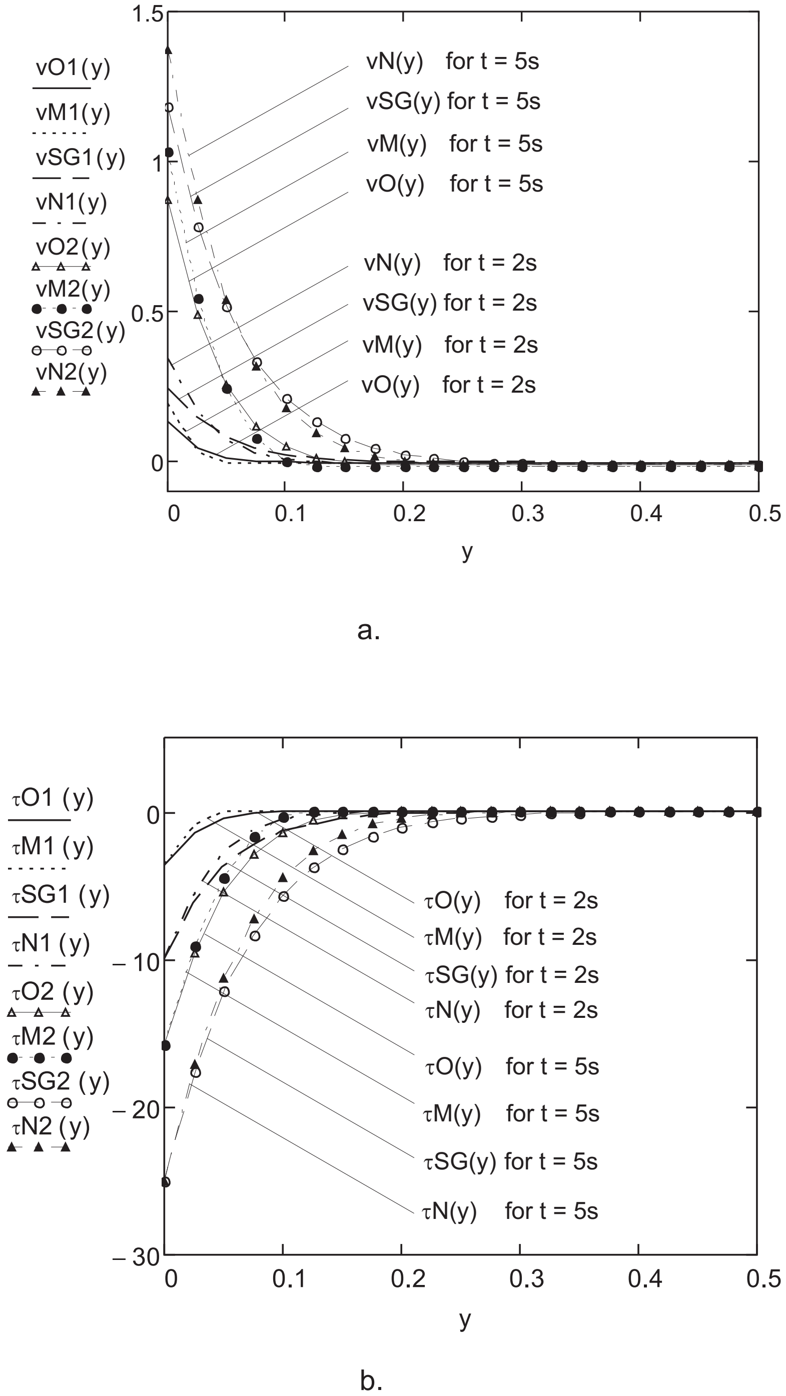
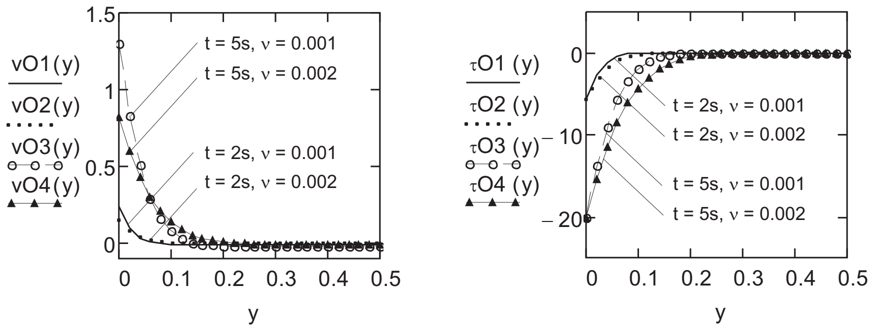
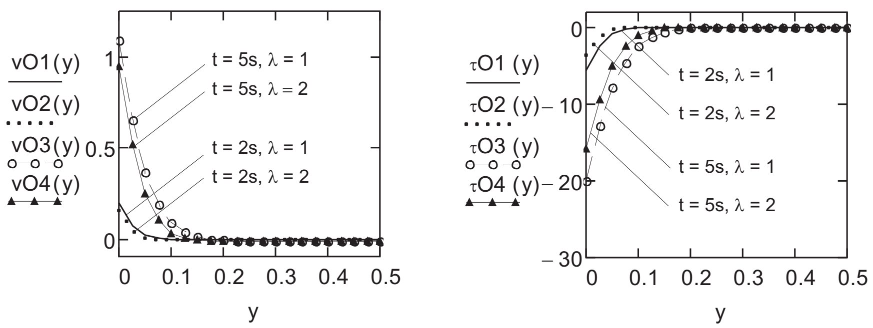


Fig.1. Profiles of the velocity v(y,t) and shear stress τ(y,t) corresponding to Oldroyd-B, Maxwell, second grade and Newtonian fluids, for $\nu = 0.001188$, $\mu = 1.045$, $\lambda = 2$ and $\lambda_r = 1$.



a. b.

Fig. 2. Profiles of the velocity v(y,t) given by Eq. (14) and shear stress τ(y,t) given by Eq. (16), for $\lambda = 1$, $\lambda_r = 0.5$ and different values of ν.



a. b.

Fig. 3. Profiles of the velocity v(y,t) given by Eq. (14) and shear stress τ(y,t) given by Eq. (16), for $\nu = 0.001188$, $\mu = 1.045$, $\lambda_r = 0.5$ and different values of λ.

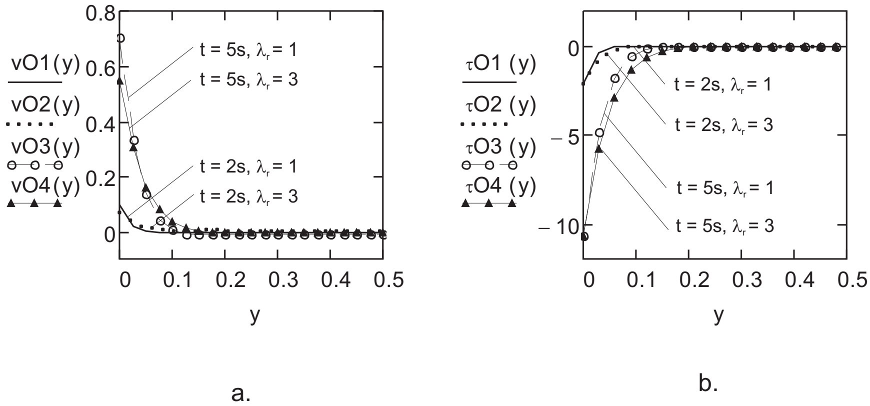


Fig. 4. Profiles of the velocity v(y,t) given by Eq. (14) and shear stress $\tau(y,t)$ given by Eq. (16), for $\nu = 0.001188$, $\mu = 1.045$, $\lambda = 4$ and different values of λ_r.