Visualization of Constrained Data Using Trigonometric Splines

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Abstract

Constructing a curve that incorporates the intrinsic feature of data is of immense importance in the realms of data visualization, computer graphics and computer aided geometric design. In this paper, most general types of positive data has been envisaged. For this purpose, a piecewise, C^1 rational cubic piecewise trigonometric function has been constrained on parameters to retain inherent data shape.

Keywords: Interpolation, Trigonometric spline, shape preservation

1. Introduction

Researchers & designers endeavoring different areas of data visualization especially computer graphics and computer aided geometric design are always in pursuit of algorithms which represent data not only efficiently but with a cutting edge to incorporate the inherent shape of data to create prototype. Positivity, convexity and monotonicity are the most common attributes of a physical process or natural phenomenon or a chemical experiment. A visual model that reflects these geometrical features of the data is high in demand.

Spline interpolating functions that yield smooth curve are prevalent in literature but incapable to retain inherent data shape. For this reason, aspiration here, is to acquire one of the hereditary attributes of data that is positivity in most general form.

Over the years, colossal amount of research has been carried out in areas of visualization and shape conservation [1-8]. In [2], authors have formulated a rational cubic piecewise function to conserve the positive trait of data. They inserted extra knots in subdivisions to retrieve positivity. Goodman et. al [9] have conserved the shape of ordered set of points pertaining on any side of the line with the help a rational cubic interpolant. Here, the authors presented two schemes; using scale factor for the weights and by inserting new interpolating point. Gregory and Sarfraz [3] developed B spline and interpolatory form of rational cubic function. The authors investigated result of deviation of tension parameter in each subdivision on shape of the curve. Lamberti and Manni [5] introduced parametric form of cubic Hermite to retain the shape of data. Here, C^2 continuity at the knots was guaranteed by establishing a system of derivative equations which is tridiagonal. Hussain and Sarfraz [6] established a rational cubic piecewise function with four shape parameters to maintain the positive shape of data. The authors utilized two of the four parameters to retrieve positivity and two were set free for shape enhancement. Schmidt and Heβ [4] preserved positivity of data by deriving conditions on derivatives at the end points of an interval of cubic polynomial. Sarfraz *et. al* [7] formulated a piecewise rational function. They preserved inherent characteristic of 2D data by setting up conditions on two of the free parameters while the remaining two were used for shape control. The conditions obtained depend on data set and the interpolating scheme was of order $O(h^3)$. Sarfraz *et. al* [8] constructed GC^1quadratic trigonometric spline with three parameters to preserve the constrained , positive, monotone and convex shape of data. Here, the authors derived conditions on two of free parameters to retain hereditary attribute of data while the remaining one parameter is set free for desired modification of the visual graphic obtained.

The subsequent sections of the paper have been organized in such a fashion that Section 2 offers a brief review of the rational cubic piecewise trigonometric function to be controlled for shape conservation. Shape parameters have been constrained to retain the intrinsic nature of data in Section 3. In section 4, proposed algorithm has been

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testified with the help of examples. Section 5 summarizes the findings of research.

2. Rational Cubic Piecewise Trigonometric Function

The rational cubic piecewise trigonometric function [1] to be used for shape preservation has been explained in this section.

Let the given data set defined over the interval $[\varphi_1, \varphi_2]$ be $\{(\xi_i, \zeta_i), i = 0,1,2,\dots,n\}$ where $\varphi_1 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = \varphi_2$. Rational cubic piecewise trigonometric function is defined over each subdivision $\chi_i = [\xi_i, \xi_{i+1}]$ as

$$S_i(\xi) = \frac{p_i(\eta)}{q_i(\eta)} \qquad (1)$$

with

$$p_i(\eta) = \alpha_i\zeta_i(1-\sin(\eta))^3 + \left\{\beta_i\zeta_i + \frac{2h_i\alpha_i d_i}{\pi}\right\}\sin(\eta)\,(1-\sin(\eta))^2 + \left\{\gamma_i\zeta_{i+1} - \frac{2h_i\delta_i d_{i+1}}{\pi}\right\}\cos(\eta)(1-\cos(\eta))^2 + \delta_i\zeta_{i+1}(1-\cos(\eta))^3,$$

$$q_i(\eta) = \alpha_i(1-\sin(\eta))^3 + \beta_i\sin(\eta)\,(1-\sin(\eta))^2 + \gamma_i\cos(\eta)\alpha_i(1-\cos(\eta))^2 + \delta_i(1-\cos(\eta))^3,$$

where

$$\eta = \frac{\pi}{2}\left(\frac{\xi-\xi_i}{h_i}\right),\ h_i = \xi_{i+1} - \xi_i.$$

The rational cubic piecewise trigonometric function (1) has the following properties:

$$S(\xi_i) = \zeta_i\,,\ S(\xi_{i+1}) = \zeta_{i+1}\,,\ S^{(1)}(\xi_i) = d_i$$
$$S^{(1)}(\xi_{i+1}) = d_{i+1}\ .$$

Here $S^{(1)}$ and d_i are the derivatives with respect to w at knots w_i. Further, arithmetic mean method has been used to calculate $d_i's$ whenever these are not provided with data set. $S(\xi) \in [\xi_0, \xi_n]$ has α_i and δ_i as free parameters.

3. Constrained Data Visualization

The rational cubic piecewise trigonometric function, reviewed in Section 2, facilitates the user to modify the shape with the help of four shape parameters. In this section, constraints on two of the shape parameters are derived so that positivity in its most general form is retained. The range of positive data is wide and it may not just lie above the line $\zeta(\xi) = 0$, but over any arbitrary line $\zeta(\xi) = m\xi + c$. Three categories of such a data has been discussed here and in each case two parameters are set free to enhance smoothness and two are utilized to keep the curve intact with the nature of positive data.

Consider the data points $\{(\xi_i, \zeta_i)\,; i = 0,\ 1, 2, \dots, n\}$ that may lie above any arbitrary straight line $\zeta(\xi) = m\xi + c$ that is $\zeta_i > m\xi_i + c\,, \forall\, i = 0, 1, 2, \dots, n$. Hence, for the curve to lie above the line

$$S(\xi) \cong S_i(\xi) > \zeta_i > m\xi_i + c \qquad (2)$$

Also m is assumed to be positive (for $m < 0$, similar approach can be adopted). For each subdivision $\chi_i = [\xi_i, \xi_{i+1}]$, $mw_i + c$ can be written as:

$$L = \varphi_i\left(1 - \frac{2}{\pi}\eta\right) + \rho_i\frac{2}{\pi}\eta, \quad \varphi_i = m\xi_i + c\ ,$$

$$\rho_i = m\xi_{i+1} + c\ . \qquad (3)$$

Equation (2) can be rewritten as

$$S_i(\xi) = \frac{p_i(\eta)}{q_i(\eta)} > L \text{ or } p_i(\eta) - Lq_i(\eta) > 0$$

which yields the following conditions on the parameter

$$\alpha_i > 0, \beta_i > \left\{0, -\frac{2h_i\alpha_i d_i}{\pi(\zeta_i - l)}\right\},$$
$$\gamma_i > \left\{0, \frac{2h_i\delta_i d_{i+1}}{\pi(\zeta_{i+1} - l)}\right\}, \delta_i > 0.$$

The above result can be briefed as:

Theorem 1. The rational cubic piecewise trigonometric function (1) remains above the given straight line, if in each subdivision $\chi_i = [\xi_i, \xi_{i+1}]$, the parameters β_i and γ_i fulfills following conditions:

$$\beta_i > max\left\{0, -\frac{2h_i\alpha_i d_i}{\pi(\zeta_i - l)}\right\}\ ;\ \alpha_i > 0,$$
$$\gamma_i > max\left\{0, \frac{2h_i\delta_i d_{i+1}}{\pi(\zeta_{i+1} - l)}\right\}\ ;\ \delta_i > 0.$$

The above conditions can be rearranged as:

$\beta_i = a_i + max\left\{0, -\frac{2h_i\alpha_i d_i}{\pi(\zeta_i - l)}\right\}$; $\alpha_i > 0$, for any real number $a_i > 0$,

$\gamma_i = b_i + max\left\{0, \frac{2h_i\delta_i d_{i+1}}{\pi(\zeta_{i+1} - l)}\right\}$; $\delta_i > 0$, for any real number $b_i > 0$.

Now if the data points $\{(\xi_i, \zeta_i)\,; i = 0,\ 1, 2, \dots, n\}$ lie below any arbitrary straight line $\zeta(\xi) = m\xi + c$ that is $\zeta_i < m\xi_i + c\,, \forall\, i = 0, 1, 2, \dots, n$. Hence, for the curve to lie below the line

$$S(\xi) \cong S_i(\xi) < \varpi_i < m\xi_i + c$$

$$L = \varphi_i\left(1 - \frac{2}{\pi}\eta\right) + \rho_i\frac{2}{\pi}\eta, \quad \varphi_i = m\xi_i + c\,,$$

$$\rho_i = m\xi_{i+1} + c\ .$$

Equation (2) can be rewritten as
$S_i(w) = \frac{p_i(\eta)}{q_i(\eta)} < L$ or $Lq_i(\eta) - p_i(\eta) > 0$
which yields the following conditions on the parameter

$$\alpha_i > 0, \beta_i > \left\{0, \frac{2h_i\alpha_i d_i}{\pi(l-\zeta_i)}\right\},$$
$$\gamma_i > \left\{0, -\frac{2h_i\delta_i d_{i+1}}{\pi(l-\zeta_{i+1})}\right\}, \delta_i > 0.$$

The above result can be stated as:

Theorem 2. The rational cubic piecewise trigonometric function (1) reside under the given straight line, if in each subdivision $\chi_i = [\xi_i, \xi_{i+1}]$, the parameters β_i and γ_i observe the following conditions:

$$\beta_i > max\left\{0, \frac{2h_i\alpha_i d_i}{\pi(l-\zeta_i)}\right\};\ \alpha_i > 0,$$
$$\gamma_i > max\left\{0, -\frac{2h_i\delta_i d_{i+1}}{\pi(l-\zeta_{i+1})}\right\};\ \delta_i > 0.$$

The above conditions can be reordered as:
$\beta_i = c_i + max\left\{0, -\frac{2h_i\alpha_i d_i}{\pi(\zeta_i - l)}\right\}$; $\alpha_i > 0$, for any real number $c_i > 0$,
$\gamma_i = d_i + max\left\{0, -\frac{2h_i\delta_i d_{i+1}}{\pi(\zeta_{i+1} - l)}\right\}$; $\delta_i > 0$, for any real number $d_i > 0$.
Now if the data points $\{(\xi_i, \zeta_i)\,; i = 0, 1, 2, \dots, n\}$ lie in between the straight line $\zeta(\xi) = m_1\xi + c_1$ that and $g(\xi) = m_2\xi + c_2, \forall\, i = 0, 1, 2, \dots, n$. Hence, for the curve to lie in between straight lines
$L_1 < S_i(\xi) < L_2, \forall\, i = 0, 1, 2, \dots, n-1.$
with
$L_1 = \varphi_i\left(1 - \frac{2}{\pi}\eta\right) + \rho_i\frac{2}{\pi}\eta, \varphi_i = m_1\xi_i + c_1$,
$\rho_i = m_1\xi_{i+1} + c_1$ and
$L_1 = \mu_i\left(1 - \frac{2}{\pi}\eta\right) + \tau_i\frac{2}{\pi}\eta, \mu_i = m_2\xi_i + c_2$,
$\tau_i = m_2\xi_{i+1} + c_2$.
which produces the following conditions on the parameters
$\alpha_i > 0, \beta_i > \left\{0, -\frac{2h_i\alpha_i d_i}{\pi(\zeta_i - l)}\right\}$,
$\beta_i > \left\{0, \frac{2h_i\alpha_i d_i}{\pi(l-\zeta_i)}\right\}$,
$\gamma_i > \left\{0, \frac{2h_i\delta_i d_{i+1}}{\pi(\zeta_{i+1} - l)}\right\}, \gamma_i > \left\{0, -\frac{2h_i\delta_i d_{i+1}}{\pi(l-\zeta_{i+1})}\right\}$,
$\delta_i > 0$.
The above discussion leads to the following result:

Theorem 3.
The rational cubic piecewise trigonometric function (1) resides between the two given straight line, if in each subdivision $\chi_i = [\xi_i, \xi_{i+1}]$, the parameters β_i and γ_i confirm following conditions:

$\beta_i > max\left\{0, -\frac{2h_i\alpha_i d_i}{\pi(\zeta_i - l)}, \frac{2h_i\alpha_i d_i}{\pi(l-\zeta_i)}\right\}, \alpha_i > 0.$
$\gamma_i > max\left\{0, \frac{2h_i\delta_i d_{i+1}}{\pi(\zeta_{i+1} - l)}, -\frac{2h_i\delta_i d_{i+1}}{\pi(l-\zeta_{i+1})}\right\}, \delta_i > 0.$
The above conditions can be rearranged as
$\beta_i = e_i + max\left\{0, -\frac{2h_i\alpha_i d_i}{\pi(\zeta_i - l)}, \frac{2h_i\alpha_i d_i}{\pi(l-\zeta_i)}\right\}, \alpha_i, e_i > 0.$
$\gamma_i = f_i + max\left\{0, \frac{2h_i\delta_i d_{i+1}}{\pi(\zeta_{i+1} - l)}, -\frac{2h_i\delta_i d_{i+1}}{\pi(l-\zeta_{i+1})}\right\}, \delta_i, f_i > 0.$

4. Demonstration This section exhibits numerical examples to implement shape preserving algorithm derived in Section 3. Table I encloses data set that lie above the straight line $y = 0.2x + 0.3$. Arbitrary assignment of real numbers to parameters in the specification of rational cubic trigonometric function fails to keep the said characteristic of data intact in visualization (Figure 1). Figure 2 is generated by applying Theorem 1 which conserves the inherent shape of data. Smoothness of Figure 2 is enhanced by choosing random values for free parameters and result is shown in Figure 3. Table II encapsulates data that lies under the straight line $y = x + 2.5$. Graphical results displayed in Figure 4 have been obtained by plugging random values of parameters in (1). Figure 4 lapses in retaining intrinsic nature of data. This drawback has been removed by applying Theorem 2 and result is shown in Figure 5. These results can further be customized according to user's required level of smoothness as shown in Figure 6. Data in Table III is constrained between two arbitrary straight lines that is above the line $y = 0.2x + 0.3$ and under the line $y = 2x + 3$. Here again the curve obtained in Figure 7 by assigning random values to shape parameters fails to retain the intrinsic characteristic of data. This loss is dealt by implementing Theorem 3 and result is shown in Figure 8. Figure 9 modifies the shape of curve by assigning random values to free parameters.

Table I

x	1	3	4	5
f	17	2.5	3	17

Table II

x	0	0.4	2	3	5
f	1.3	0.7	2	5.1	0.5

Table III

x	1	3	4	5
f	17	2.5	3	17

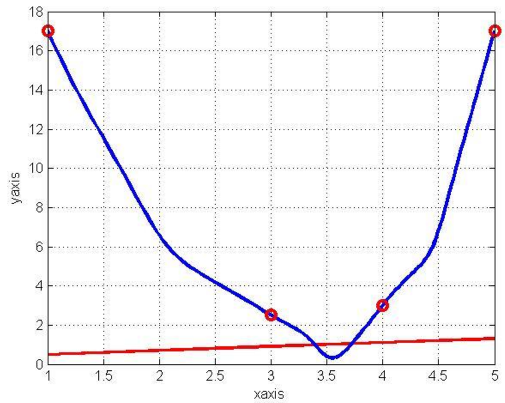

Figure 1

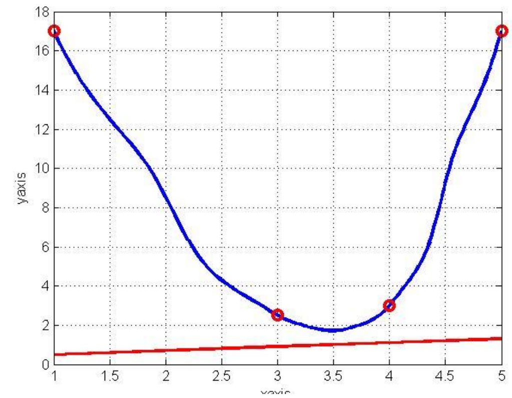

Figure 2

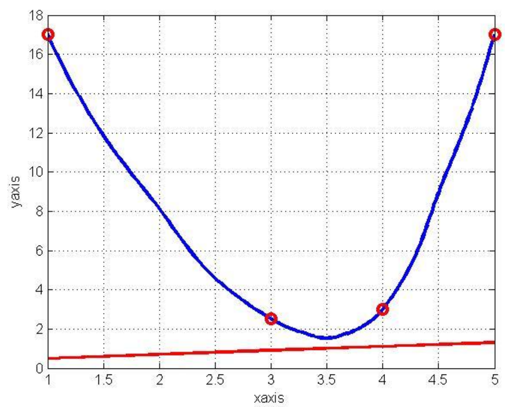

Figure 3

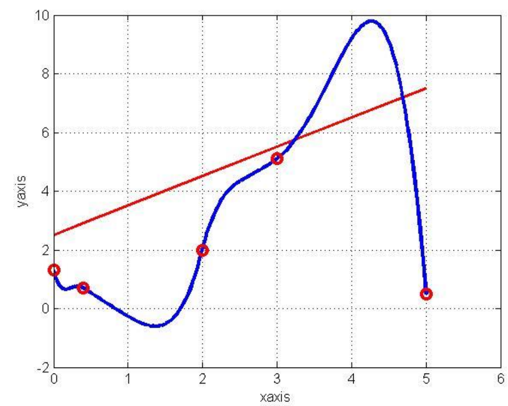

Figure 4

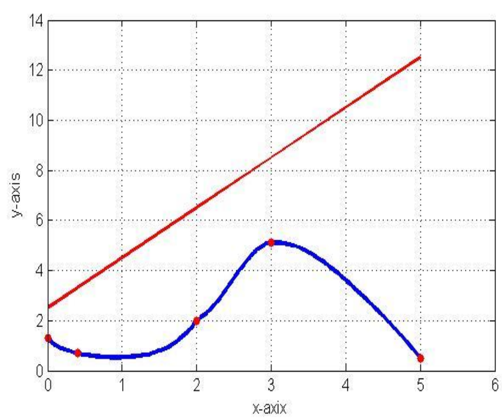

Figure 5

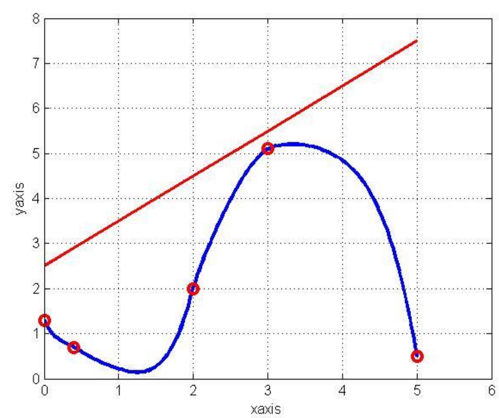

Figure 6

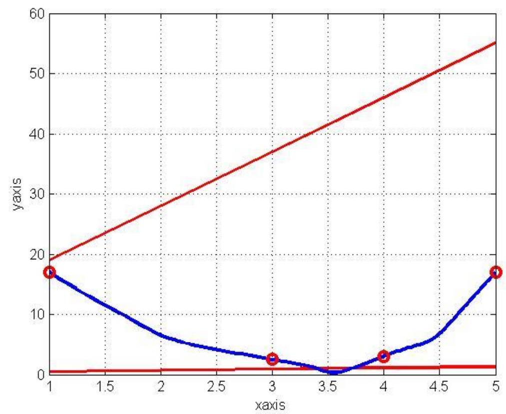

Figure 7

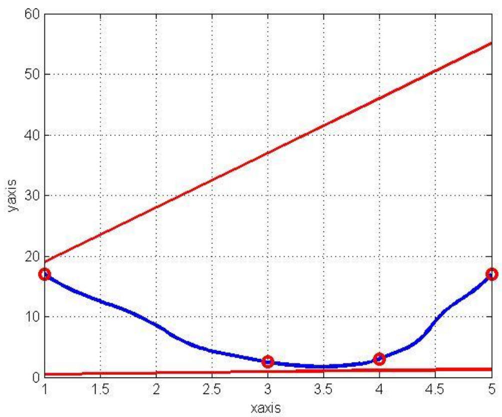

Figure 8

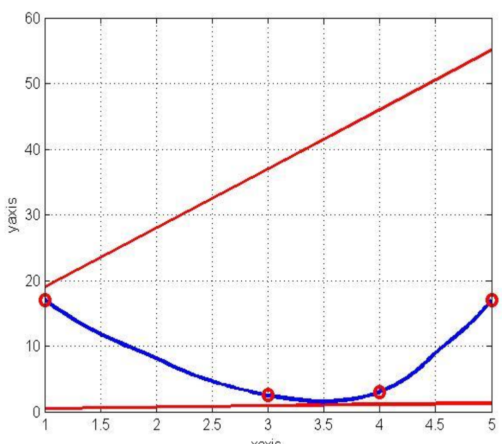

Figure 9

5. Conclusion

This paper takes the rational trigonometric function into account to conserve the shape of positive data when it lies above or below any arbitrary straight line. The derivatives at the knots are calculated by geometric mean method. The proposed algorithm offers a variety of advantageous features. It works well for uniform as well as non-uniform data points. There is no extra knot inserted to preserve the inherent characteristic of data. Derivatives of trigonometric functions being non terminating in nature, undertakes higher order smoothness even with lower degree trigonometric function. Also, the orthogonality of trigonometric functions sets off much smoother results than that of algebraic functions.

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