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Generation of zonal flows by coupled electrostatic drift and ion-acoustic waves

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Generation of sheared zonal flow by low-frequency coupled electrostatic drift and ion-acoustic waves is presented. Primary waves of different (small, intermediate, and large) scales are considered, and the appropriate system of equations consisting of generalized Hasegawa-Mima equation for the electrostatic potential (involving both vector and scalar nonlinearities) and equation of parallel to magnetic field ions motion is obtained. It is shown that along with the mean poloidal flow with strong variation in minor radius mean sheared toroidal flow can also be generated. According to laboratory plasma experiments, main attention to large scale drift-ion-acoustic wave is given. Peculiarities of the Korteweg-de Vries type scalar nonlinearity due to the electrons temperature non-homogeneity in the formation of zonal flow by large-scale turbulence are widely discussed. Namely, it is observed that such type of flows need no generation condition and can be spontaneously excited. *Published by AIP Publishing*. [http://dx.doi.org/10.1063/1.4989708]

I. INTRODUCTION

Drift waves play a decisive role in the magnetic trapping of plasma. Drift waves were predicted by Rudakov and Sagdeev[1] and later Mikhailovskii[2–4] has contributed to understanding of the problem. A comprehensive survey of the drift waves turbulence and associated anomalous transport was elucidated by Horton.[5] Nonlinear dynamics of drift waves in plasmas is primarily described by the classical Hasegawa-Mima (HM) equation,[6,7] providing different solutions which involve turbulent, coherent, and wave behaviors. It should be noted that the nonlinear term in the standard HM equation has the structure of type $J(a,b) = [\nabla a \times \nabla b]_z$, where a and b are certain functions of wave field. Such nonlinearity is called vector nonlinearity and provides the existence of dipolar nonlinear structures. On the importance of other, so called scalar, Korteweg-de Vries (KdV) type nonlinearity $\propto \varphi^2$ in the nonlinear theory of drift waves was indicated by Petviashvili.[8] Scalar nonlinearities are responsible for the existence of monopolar nonlinear structures. Simultaneous accounting of both vector and scalar nonlinearities first was performed by Petviashvili[9] when investigating the problem of Jovian Great Red Spot. The comprehensive analysis of both (monopolar and dipolar) types of drift vortical structures was given by Mikhailovskii.[10] Later, Nezlin and Chernikov[11] elucidated the new localizing role of vector and scalar nonlinearities in the process of formation of solitary nonlinear structures and emphasized that depending on the wavelengths scale drift waves turbulence should be described by the more complex, so called generalized HM equation.

The other problem which is closely connected with drift wave turbulence is the generation of sheared zonal flow spontaneously arising in laboratory plasmas as a consequence of the secondary instability of plasma due to the nonlinear interaction between the primary oscillations. The nonlinear vector and scalar terms in the equations permit us to consider a three-wave interaction, in which the coupling between the pump electrostatic drift-ion-acoustic waves and side-band modes generates large-scale modes, so called zonal flows. Actually, the zonal flow is spontaneously generated from small-scale drift wave fluctuations via the action of Reynolds stresses. This problem have been attracted a scientific attention because according to the acceptable statement sheared zonal flow suppresses plasma turbulence and reduces the anomalous transport of heat and particles across the magnetic surfaces due to the energy transport toward large scale structures as a result of the inverse energy cascade. Basically carried out investigations on sheared zonal flow generation problem by the electrostatic drift waves can be divided into two classes. The first one invokes the classical coherent parametric instability method to study the generation of sheared zonal flow,[12–17] whereas the second class uses the representation of electrostatic drift waves by a wave-kinetic equation coupled to the zonal flow equation.[18,19] The new methodical achievement was developed by Mikhailovskii *et al.*,[20] where the parametric approach was modified assuming the spectrum of primary modes to be arbitrary (instead of monochromatic consideration).

In the present work, we consider the possibility of generation of zonal flow on the low-frequency coupled electrostatic drift and ion-acoustic waves. As in the tokamak plasma experiments[5,21,22] mainly large-scale drift waves ($k_\perp \rho_s \leq 1$, where ρ_s is the ion Larmor radius defined at the electron temperature) are observed we will draw our attention to the large-scale solitary structures and derive the generalized HM equation for the coupled drift-ion-acoustic waves. In Sec. II, a system of basic equations comprising the general HM equation for electrostatic potential and equation describing parallel to magnetic field ions motion valid for arbitrary wavelengths of primary waves is obtained. In Sec. III, a linear regime is considered in detail. In Sec. IV a

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system of basic nonlinear equations (obtained in Sec. II) is separately considered in accordance with the wavelengths (small, intermediate, and large scales) of primary waves. In Sec. V, we consider the possibility of sheared zonal flow generation by coupled electrostatic drift-ion-acoustic waves. In Sec. VI, we discuss obtained results.

II. BASIC EQUATIONS

We consider electrostatic low-frequency waves with the frequency much smaller than the ion cyclotron frequency (i.e., $\omega \ll \omega_{ci}$) in a magnetized (with the magnetic field $B\mathbf{e}_z$) and inhomogeneous (with the density $n_0(x)$ and temperature $T_e(x)$) plasma. The linear waves in the form of drift and ion-acoustic waves are known to exist in such a plasma if the phase velocity in the direction of the magnetic field, ω/k_z is between the electron and the ion thermal velocities, v_{Te} and v_{Ti}.

The equations needed for a more complete description of low-frequency electrostatic drift and ion-acoustic waves in plasmas are the equations of motion and continuity for the ions, and the Boltzmann distribution of the electrons

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{e}{m}\nabla\varphi + \omega_{ci}\mathbf{v}\times\mathbf{e}_z, \tag{1}$$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0, \tag{2}$$

$$n = n_0(x)\exp\left(\frac{e\varphi}{T_e(x)}\right). \tag{3}$$

Here, n, $\mathbf{v}$, e, and m are the ion density, velocity, charge, and mass, respectively, $\varphi\,(t,x,y,z)$ is the electrostatic potential, and $\omega_{ci} = eB/m$ is the ion cyclotron frequency. We assume the plasma to be quasineutral, so that $n_e = n$. The magnetic field $\mathbf{B} = B\mathbf{e}_z$ is assumed to be constant and homogeneous, and $T_e \gg T_i$, which means that we can neglect the ion pressure in the equation of motion. The equilibrium density $n_0(x)$ and electron temperature $T_e(x)$ are both assumed to be inhomogeneous in the x direction.

According to Eq. (3), electrons attain thermal equilibrium along the magnetic field lines, so we must require that the phase velocity of the electrostatic perturbations along the magnetic field $\omega/k_z \ll \mathrm{v}_{Te}$. It also has to be smaller than the Alfven velocity $c_A = B/(\mu_0 n_0 m)^{1/2}$ which means that the magnetic field perturbations due to the parallel current can be neglected. Hence, k_z must be finite.

We consider the coupling of drift waves with ion-acoustic ones and assume the weak z dependence of the fields. Taking the *curl* of Eq. (1) and using Eq. (2), we get the following "freezing-in field equation":[23]

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\left(\frac{\mathbf{e}_z\omega_{ci} + \boldsymbol{\Omega}}{n}\right) = \left(\frac{\mathbf{e}_z\omega_{ci} + \boldsymbol{\Omega}}{n}\cdot\nabla\right)\mathbf{v}, \tag{4}$$

where $\boldsymbol{\Omega} = \nabla\times\mathbf{v}$ is the vorticity. The obtained equation is valid for the 3D perturbations, and the new term on the right-hand side describes vortex stretching (cf. Ref. 16).

Further, we will use the small expansion parameter ε,

$$\varepsilon \sim \frac{1}{\omega_{ci}}\frac{\partial}{\partial t} \sim \frac{1}{k_z \mathrm{v}_{Te}}\frac{\partial}{\partial t} \sim \frac{\Omega}{\omega_{ci}} \sim \frac{e\varphi}{T_e} \sim \frac{L}{L_{n,T}} \ll 1, \tag{5}$$

where L is the typical length scale of the fluctuations, and $L_{n,T}$ is the inhomogeneity scale of the equilibrium density and temperature, respectively. The characteristic wave dispersion scale length is $\rho_s = (T_e/m\omega_{ci}^2)^{1/2}$, which represents the ion Larmor radius defined at the electron temperature T_e.

To express Eq. (4) in terms of potential $\varphi\,(t,x,y,z)$, we represent the total particle velocity as $\mathbf{v} = \mathbf{v}_\perp + \mathbf{e}_z w$. For low-frequency waves $\omega \ll \omega_{ci}$, Eq. (1) implies[10]

$$\mathbf{v}_\perp = \mathbf{v}_E + \mathbf{v}_I\,, \tag{6}$$

where $\mathbf{v}_E$ is the electric drift velocity (or cross field drift velocity) defined as

$$\mathbf{v}_E = \frac{1}{B}\mathbf{e}_z\times\nabla_\perp\varphi = \frac{1}{B}\mathbf{E}\times\mathbf{e}_z, \tag{7}$$

and $\mathbf{v}_I$ is the inertial part of the transverse velocity

$$\mathbf{v}_I = \frac{1}{\omega_{ci}}\mathbf{e}_z\times\frac{d_0}{dt}\mathbf{v}_E, \tag{8}$$

where $d_0/dt = \partial/\partial t + \mathbf{v}_E\cdot\nabla + w\,\partial/\partial z$.

Taking into account the conditions (5) and substituting Eqs. (6)–(8) into the z component of Eq. (4), we get the following expression for the electrostatic potential:

$$\begin{aligned}&\frac{\partial\varphi}{\partial t} - \rho_s^2\frac{\partial}{\partial t}\Delta_\perp\varphi - \rho_s^2\omega_{ci}\frac{1}{n_0}\frac{dn_0}{dx}\frac{\partial\varphi}{\partial y} + \rho_s^2\omega_{ci}\frac{1}{T_e}\\ &\times\frac{dT_e}{dx}\varphi\frac{\partial\varphi}{\partial y} - \rho_s^4\omega_{ci}J(\varphi,\Delta_\perp\varphi) + w\frac{\partial\varphi}{\partial z} + \frac{\partial w}{\partial z}\\ &-\rho_s^2 w\frac{\partial}{\partial z}\Delta_\perp\varphi + \rho_s^2\Delta_\perp\varphi\frac{\partial w}{\partial z} - \rho_s^2\frac{\partial w}{\partial x}\frac{\partial^2\varphi}{\partial x\partial z}\\ &-\rho_s^2\frac{\partial w}{\partial y}\frac{\partial^2\varphi}{\partial y\partial z} = 0\,.\end{aligned} \tag{9}$$

Other equation describing the parallel to magnetic field ions motion, we will get from the z component of the equation of motion (1) as

$$\frac{\partial w}{\partial t} + \rho_s^2\omega_{ci}J(\varphi,w) + w\frac{\partial w}{\partial z} = -\mathrm{v}_s^2\frac{\partial\varphi}{\partial z}. \tag{10}$$

Here, $\mathrm{v}_s = (T_e/m)^{1/2}$ is the ion-acoustic speed and the Jacobian $J(a,b) = \partial_x a\partial_y b - \partial_y a\partial_x b$, and $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplacian. Equations (9) and (10) compose the initial closed system of equations valid for arbitrary $k_\perp\rho_s$. In the both equations, potential φ is normalized by T_e/e. As to the ions parallel motion Eq. (10), it contains both vector and scalar nonlinearities.

Compared to the classical HM equation with respect to drift waves, the generalized Eq. (9) contains additional new scalar nonlinearity of KdV type: $\rho_s^2\omega_{ci}\frac{1}{T_e}\frac{dT_e}{dx}\varphi\frac{\partial\varphi}{\partial y}$. The standard HM equation containing only the vector nonlinearity

$\rho_s^4\omega_{ci}J(\varphi,\Delta_\perp\varphi)$ is valid only for the small-scale structures when the characteristic size $L \le \rho_s$ and predicts the existence only of dipolar vortices (cyclone-anticyclone pairs). Solitary monopole type vortices (i.e., either cyclones or anticyclones) can be described only by the generalized HM equation of type (9) containing scalar nonlinearities. Monopole type solitary structures were first observed in laboratory modeling of solitary Rossby vortices.[24] Kaladze *et al.*[25] showed numerically that the presence of the scalar nonlinearity plays the role of instability forming monopole vortical structures of definite polarity as a result of breaking large-scale dipole ones. The generalized HM equation for the electrostatic drift waves in connection with the zonal flow generation was obtained by Kaladze *et al.*[16] Dynamics of large-scale drift vortical structures in electron-positron-ion plasmas was discussed in Ref. 26. Generation of zonal flows by electrostatic drift waves of arbitrary wavelength size in electron-positron-ion plasmas was considered in Ref. 27. Generation of large-scale zonal flows by the small-scale electrostatic drift wave turbulence in the magnetized plasma under the action of mean poloidal sheared flow was discussed by Kaladze and Kharshiladze in recently appeared paper.[28]

III. LINEAR REGIME

In the linear regime from Eqs. (9) and (10), we get the following system of equations:

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} - \rho_s^2\dfrac{\partial}{\partial t}\Delta_\perp\varphi - \rho_s^2\omega_{ci}\dfrac{1}{n_0}\dfrac{dn_0}{dx}\dfrac{\partial\varphi}{\partial y} + \dfrac{\partial w}{\partial z} = 0\,, \\ \dfrac{\partial w}{\partial t} = -\mathrm{v}_s^2\dfrac{\partial\varphi}{\partial z}\,. \end{cases} \tag{11}$$

Derivation of the first equation over t and usage of the second one gives the following coupled drift-ion acoustic waves equation:

$$\frac{\partial^2\varphi}{\partial t^2} - \rho_s^2\frac{\partial^2}{\partial t^2}\Delta_\perp\varphi - \rho_s^2\omega_{ci}\frac{1}{n_0}\frac{dn_0}{dx}\frac{\partial^2\varphi}{\partial t\partial y} - \mathrm{v}_s^2\frac{\partial^2\varphi}{\partial z^2} = 0\,. \tag{12}$$

In $(\mathbf{k},\,\omega)$ space, we get the following appropriate algebraic equation:

$$\omega^2(1+k_\perp^2\rho_s^2) - \omega\,k_y\rho_s^2\beta_n\omega_{ci} - k_z^2\mathrm{v}_s^2 = 0\,, \tag{13}$$

where $\beta_n = -\frac{1}{n_0}\frac{dn_0}{dx} > 0$. The roots of this equation are given as follows:

$$\omega_{1,2} = \frac{k_y\rho_s^2\omega_{ci}\beta_n \pm \sqrt{k_y^2\rho_s^4\omega_{ci}^2\beta_n^2 + 4k_z^2\mathrm{v}_s^2\left(1+k_\perp^2\rho_s^2\right)}}{2\left(1+k_\perp^2\rho_s^2\right)}. \tag{14}$$

Equation (14) defines fast ω_1 and slow ω_2 coupled drift ion-acoustic waves.

As to the generated wave frequencies from Eq. (14), we get the following expressions for the linear phase velocities:

$$\left(\frac{\omega}{k_y}\right)_{1,2} = \frac{\mathrm{v}^*}{2\left(1+k_\perp^2\rho_s^2\right)}\left[1\pm\sqrt{1+4\frac{k_z^2}{k_y^2}\frac{1}{\rho_s^2\beta_n^2}\left(1+k_\perp^2\rho_s^2\right)}\right]. \tag{15}$$

Here, $\mathrm{v}^* = \beta_n\rho_s^2\omega_{ci}$ is diamagnetic drift velocity calculated at the electron temperature. Let us consider the following particular cases:

(a) In case of $k_z = 0$, we have the single drift waves

$$\omega_1 = \frac{k_y\mathrm{v}^*}{1+k_\perp^2\rho_s^2}. \tag{16}$$

(b) In case of $k_y = 0$ ($\beta_n = 0$), we have the single ion-acoustic waves

$$\omega_{1,2} = \pm\frac{k_z\mathrm{v}_s}{\sqrt{1+k_x^2\rho_s^2}}. \tag{17}$$

Note that owing to the coupling with drift waves ion-acoustic waves become dispersive.

(c) In the case of sufficiently small longitudinal wave numbers $k_z \ll k_y$, we have the following mixed frequencies:

$$\omega_1 = \frac{k_y\mathrm{v}^*}{1+k_\perp^2\rho_s^2}\left(1+\frac{k_z^2\mathrm{v}_s^2\left(1+k_\perp^2\rho_s^2\right)}{k_y^2\mathrm{v}^{*2}}\right), \quad \omega_2 = -\frac{k_z^2\mathrm{v}_s^2}{k_y\mathrm{v}^*}. \tag{18}$$

Here, $\omega_{1,2}$ corresponds to the upper and bottom signs, respectively, in Eq. (15).

(d) In the case of sufficiently small $k_y \ll k_z$, we get

$$\begin{aligned} \omega_1 &= \frac{k_z\mathrm{v}_s}{\sqrt{1+k_\perp^2\rho_s^2}}\left(1+\frac{k_y\mathrm{v}^*}{2k_z\mathrm{v}_s\sqrt{1+k_\perp^2\rho_s^2}}\right), \\ \omega_2 &= -\frac{k_z\mathrm{v}_s}{\sqrt{1+k_\perp^2\rho_s^2}}\left(1-\frac{k_y\mathrm{v}^*}{2k_z\mathrm{v}_s\sqrt{1+k_\perp^2\rho_s^2}}\right). \end{aligned} \tag{19}$$

IV. NONLINEAR REGIME

In this section, we will consider system of Eqs. (9) and (10) for different scales of wavelengths and write down the appropriate nonlinear equations.

A. Small and intermediate wavelengths $k_\perp\rho_s \ge 1$

Using the following estimations:

$$\omega \sim k_y\left|\frac{1}{n_0}\frac{dn_0}{dx}\right|\rho_s^2\omega_{ci} \sim k_\perp^2\rho_s^2\omega_{ci}\frac{L}{L_n} \sim \omega_{ci}\frac{L}{L_n} \sim k_z\mathrm{v}_s, \tag{20}$$

we find $k_z \sim 1/L_n$. Further comparing the first terms of both sides in Eq. (10), we find $w \sim k_z\varphi\mathrm{v}_s^2/\omega$. Substituting here $\omega \approx k_z\mathrm{v}_s$, we get the estimation

$$\omega\,\varphi \sim k_z w \sim \frac{w}{L_n}. \tag{21}$$

Under the conditions (20), (21), we get the following system of simplified initial equations of the basic Eqs. (9) and (10) (cf. Ref. 23):

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} - \rho_s^2 \dfrac{\partial}{\partial t}\Delta_\perp\varphi + \rho_s^2\omega_{ci}\beta_n\dfrac{\partial\varphi}{\partial y} - \rho_s^4\omega_{ci}J(\varphi,\Delta_\perp\varphi) + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial w}{\partial t} + \rho_s^2\omega_{ci}J(\varphi,w) = -\mathrm{v}_s^2\dfrac{\partial\varphi}{\partial z}. \end{cases} \tag{22}$$

B. Large-scale wavelengths $k_\perp\rho_s \ll 1$

As it seen from the estimation (20), the longitudinal wave number $k_z \sim k_\perp\rho_s\frac{1}{L_n}$. Analogously to Eq. (21), we have

$$\omega\,\varphi \sim k_z w \sim \omega_{ci}k_\perp^2\rho_s^2\frac{L}{L_n}\varphi. \tag{23}$$

Under these conditions, the basic equations (9) and (10) can be reduced to the system

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} - \rho_s^2\dfrac{\partial}{\partial t}\Delta_\perp\varphi + \rho_s^2\omega_{ci}\beta_n\dfrac{\partial\varphi}{\partial y} - \rho_s^2\omega_{ci}\beta_T\varphi\dfrac{\partial\varphi}{\partial y} - \rho_s^4\omega_{ci}J(\varphi,\Delta_\perp\varphi) + w\dfrac{\partial\varphi}{\partial z} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial w}{\partial t} + \rho_s^2\omega_{ci}J(\varphi,w) + \mathrm{v}_s^2\dfrac{\partial\varphi}{\partial z} = 0, \end{cases} \tag{24}$$

where $\beta_T = -\frac{1}{T_e}\frac{dT_e}{dx} > 0$. Note that the vector and scalar nonlinearities in the Eq. (24) are equal by the order when $k_\perp^2\rho_s^2 \sim L/L_T$.

V. EXCITATION OF ZONAL FLOW

To consider the problem of zonal flow excitation by coupled electrostatic drift-ion-acoustic waves, we will follow the modified parametric method developed by Mikhailovskii *et al.*[20] for the system of equations describing drift-Alfven waves (see also Ref. 29). In addition in what follows, we normalize time by ω_{ci}^{-1}, lengths by ρ_s, and velocity w by $\rho_s\omega_{ci}$.

Obtained in Sec. IV systems, a system of dynamic nonlinear equations give the possibility to consider both the zonal flow and ion's parallel to magnetic field sheared flow generation by the coupled drift-ion-acoustic waves. The nonlinear vector and scalar nonlinearities in these equations permit us to consider a three-wave interaction, in which the coupling between the pump electrostatic drift-ion-acoustic waves and side-band modes generates large-scale modes, so called zonal flows. Since the zonal flow varies on a much longer time scale than the comparatively small-scale coupled waves, so one can use a multiple-scale expansion, assuming that there is a sufficient spectral gap separating the large- and small-scale motions. Accordingly, perturbed quantities are split in 3-components, $X = \tilde{X} + \hat{X} + \bar{X}$, where

$$\begin{cases} \tilde{X} = \sum\limits_{\mathbf{k}}\left[\tilde{X}_+(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}}t) + \tilde{X}_-(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega_{\mathbf{k}}t)\right], \\ \hat{X} = \sum\limits_{\mathbf{k}}\left[\hat{X}_+(\mathbf{k})\exp(i\mathbf{k}_+\cdot\mathbf{r} - i\omega_{\mathbf{k}_+}t) + \hat{X}_-(\mathbf{k})\exp(i\mathbf{k}_-\cdot\mathbf{r} - i\omega_{\mathbf{k}_-}t) + c.c.\right], \\ \bar{X} = \bar{X}_0(\mathbf{k})\exp(-i\Omega t + iq_x x) + c.c., \end{cases} \tag{25}$$

describe the spectrum of pump EM modes ($\tilde{X}_-(\mathbf{k}) = \tilde{X}_+^*(\mathbf{k})$, where $*$ means the complex conjugate), the sideband modes spectrum, and 1D zonal-flow mode, respectively. The following energy and momentum conservations are fulfilled between the three waves: $\omega_\pm = \Omega \pm \omega$, $\mathbf{k}_\pm = q_x\mathbf{e}_x \pm \mathbf{k}$. There exist small parameters

$$\frac{|\Omega|}{|\omega_{\mathbf{k}}|} \sim \frac{|q_x|}{|k_\perp|} \ll 1, \tag{26}$$

which are typical for the zonal flow excitation problems.

A. Small and intermediate-scale structures, $k_\perp\rho_s \geq 1$.

First, we will consider the intermediate-scale structures case which can be described by the system (22). Substituting Eq. (25) into the dimensionless system of (22) and neglecting the contribution of small nonlinear terms (like the standard quasilinear procedure), we get the following system for spectral components of the main pump drift-ion-acoustic modes:

$$\begin{cases} \omega_{\mathbf{k}}(1+k_\perp^2)\tilde{\varphi}_\pm - \beta_n k_y\tilde{\varphi}_\pm = k_z\tilde{w}_\pm, \\ \omega_{\mathbf{k}}\tilde{w}_\pm - k_z\tilde{\varphi}_\pm = 0. \end{cases} \tag{27}$$

Solving this system, we get the dispersion relation (13) in the following dimensionless form:

$$\omega_{\mathbf{k}}^2(1+k_\perp^2) - \omega_{\mathbf{k}}\beta_n k_y - k_z^2 = 0, \tag{28}$$

having the general solution

$$\omega_{\mathbf{k}1,2}=\frac{\beta_n k_y \pm\sqrt{\beta_n^2 k_y^2+4k_z^2\left(1+k_\perp^2\right)}}{2\left(1+k_\perp^2\right)}. \tag{29}$$

Equation (29) describes fast ($\omega_{\mathbf{k}1}$) and slow ($\omega_{\mathbf{k}2}$) drift-ion-acoustic waves. When $\beta_n=0$, we get the dispersive ion-acoustic wave frequencies

$$\omega_{\mathbf{k}1,2}=\pm\frac{k_z}{\left(1+k_\perp^2\right)^{1/2}}. \tag{30}$$

In the case of small $k_z\to 0$, we have the following two oscillations:

$$\omega_1=\frac{\beta_n k_y}{1+k_\perp^2},\quad \omega_2=-\frac{k_z^2}{\beta_n k_y}. \tag{31}$$

The basic system of equations describing the evolution of mean electrostatic potential and parallel to magnetic field flows can be found in the following way:[20,29]

$$\begin{cases} -i\Omega\bar{\varphi}_0=R_\perp=-\dfrac{q_x^2}{1+q_x^2}\sum\limits_{\mathbf{k}}k_y r_\perp(\mathbf{k}), \\ -i\Omega\bar{w}_0=R_{\mathrm{II}}=q_x\sum\limits_{\mathbf{k}}k_y r_{\mathrm{II}}(\mathbf{k}), \end{cases} \tag{32}$$

where

$$r_\perp(\mathbf{k})=\tilde{\varphi}_-\hat{\chi}_+-\tilde{\varphi}_+\hat{\chi}_-,\quad r_{\mathrm{II}}(\mathbf{k})=\tilde{\varphi}_-\hat{\lambda}_+-\tilde{\varphi}_+\hat{\lambda}_-. \tag{33}$$

Here, $(\Omega,\, q_x\mathbf{e}_x)$ is the frequency and wave vector of the zonal-flow modes. The right-hand sides of Eq. (32) represent the driving forces of zonal flows, which are the mean electrostatic stress ($R_\perp$) and electromotive force (R_{II}), respectively. Note that the second equation in (33) is the evolution equation of the parallel to magnetic field mean flow. Auxiliary sideband amplitudes in Eq. (33) are determined by

$$\hat{\chi}_\pm=-(q_x\pm 2k_x)\hat{\varphi}_\pm,\quad \hat{\lambda}_\pm=\hat{w}_\pm-\frac{k_z}{\omega_{\mathbf{k}}}\hat{\varphi}_\pm. \tag{34}$$

Thus, in order to determine the functions (34), sideband amplitudes should be found. These amplitudes satisfy the following system of equations:

$$\begin{cases}\left[\omega_\pm\left(1+k_{\perp\pm}^2\right)\mp\beta_n k_y\right]\hat{\varphi}_\pm\mp k_z\hat{w}_\pm=\mp i\left(q_x^2-k_\perp^2\right)k_y q_x\tilde{\varphi}_\pm\bar{\varphi}_0, \\ \pm k_z\hat{\varphi}_\pm-\omega_\pm\hat{w}_\pm=\pm i\left(\bar{w}_0-\dfrac{k_z}{\omega_{\mathbf{k}}}\bar{\varphi}_0\right)k_y q_x\tilde{\varphi}_\pm. \end{cases} \tag{35}$$

Note that energy and momentum conservation is imposed on sideband frequencies $\omega_\pm$ and wave vector $\mathbf{k}_\pm$ by requiring that $\omega_\pm=\Omega\pm\omega$ and $\mathbf{k}_\pm=q_x\mathbf{e}_x\pm\mathbf{k}$. The solution of the system (35) may be represented as

$$\begin{cases}\hat{\varphi}_\pm=i\dfrac{k_y q_x\tilde{\varphi}_\pm}{D_\pm}\left\{-k_z\bar{w}_0+\bar{\varphi}_0\left[\dfrac{k_z^2}{\omega_{\mathbf{k}}}\mp\omega_\pm\left(q_x^2-k_\perp^2\right)\right]\right\}, \\ \hat{w}_\pm=i\dfrac{k_y q_x\tilde{\varphi}_\pm}{D_\pm}\left\{\mp\bar{w}_0\left[\omega_\pm\left(1+k_{\perp\pm}^2\right)\mp\beta_n k_y\right]\pm\bar{\varphi}_0\dfrac{k_z}{\omega_{\mathbf{k}}}\left[\omega_\pm\left(1+k_{\perp\pm}^2\right)\mp\beta_n k_y\mp\omega_{\mathbf{k}}\left(q_x^2-k_\perp^2\right)\right]\right\}. \end{cases} \tag{36}$$

Here

$$D_\pm=\omega_\pm^2(1+k_{\perp\pm}^2)\mp\omega_\pm\beta_n k_y-k_z^2. \tag{37}$$

Using Eq. (36) into Eq. (34), we get

$$\begin{aligned}\hat{\lambda}_\pm=i\frac{q_x k_y}{D_\pm}\tilde{\varphi}_\pm\Bigg\{&\mp\bar{\varphi}_0\frac{\Omega}{\omega_{\mathbf{k}}}k_z\left(k_\perp^2-q_x^2\right) \\ &+\left(\bar{w}_0-\bar{\varphi}_0\frac{k_z}{\omega_{\mathbf{k}}}\right)\left[\mp\Omega\left(1+q_x^2+k_\perp^2\right)\right. \\ &\left.-2\Omega q_x k_x-\omega_{\mathbf{k}}q_x^2\mp 2\omega_{\mathbf{k}}q_x k_x\right]\Bigg\},\end{aligned} \tag{38}$$

$$\begin{aligned}\hat{\chi}_\pm=&-i\frac{k_y q_x}{D_\pm}(q_x\pm 2k_x)\tilde{\varphi}_\pm \\ &\times\left\{-k_z\bar{w}_0+\bar{\varphi}_0\left[\frac{k_z^2}{\omega_{\mathbf{k}}}\mp\omega_\pm\left(q_x^2-k_\perp^2\right)\right]\right\}.\end{aligned} \tag{39}$$

Using the superscripts "(1), (2),…" to indicate the order of magnitudes with respect to q_x and Ω, we represent Eq. (37) as

$$D_\pm=\pm D^{(1)}+D^{(2)}\pm D^{(3)}+D^{(4)}, \tag{40}$$

where

$$\begin{cases}D^{(1)}=2q_x k_x\omega_{\mathbf{k}}^2+2\Omega\omega_{\mathbf{k}}(1+k_\perp^2)-\Omega\beta_n k_y, \\ D^{(2)}=\Omega^2(1+k_\perp^2)+q_x^2\omega_{\mathbf{k}}^2+4\Omega\omega_{\mathbf{k}}q_x k_x, \\ D^{(3)}=2q_x\Omega^2 k_x+2\Omega\omega_{\mathbf{k}}q_x^2, \\ D^{(4)}=\Omega^2 q_x^2. \end{cases} \tag{41}$$

Using the expansion over the small parameters (26) and keeping only necessary main terms, we find finally for Eq. (32)

$$\begin{aligned}r_{II}(\mathbf{k})=i\frac{q_x k_y\Omega}{D^{(1)2}}I_{\mathbf{k}}\Bigg\{&\bar{\varphi}_0\frac{k_z}{\omega_{\mathbf{k}}}\Bigg[-\Omega^2\left(1+k_\perp^2\right) \\ &+q_x^2\left(\omega_{\mathbf{k}}^2\left(k_\perp^2-4k_x^2\right)+k_z^2\right)+\frac{2\Omega q_x k_x}{\omega_{\mathbf{k}}}\left(k_z^2-2\omega_{\mathbf{k}}^2\right)\Bigg] \\ &+\bar{w}_0\Bigg[\Omega^2\left(1+k_\perp^2\right)^2+q_x^2\left(4k_x^2\omega_{\mathbf{k}}^2-k_z^2\right) \\ &+2\Omega q_x k_x\left(\omega_{\mathbf{k}}\left(1+k_\perp^2\right)+\beta_n k_y\right)\Bigg]\Bigg\},\end{aligned} \tag{42}$$

$$r_\perp(\mathbf{k}) = -i\frac{q_x k_y \Omega}{D^{(1)2}} I_\mathbf{k} \Big\{ \bar{\varphi}_0 \Big[q_x \Big(\omega_\mathbf{k}^2 k_\perp^2 (1+k_\perp^2) - 4k_x^2 \omega_\mathbf{k}^2 k_\perp^2 - 8k_x^2 k_z^2 + \frac{k_z^4}{\omega_\mathbf{k}^2} + k_z^2(1+2k_\perp^2) \Big) - 2\Omega k_x \frac{k_z^2}{\omega_\mathbf{k}} \Big] + \bar{w}_0 k_z \Big[2\Omega k_x (1+k_\perp^2) + q_x \Big(-\frac{k_z^2}{\omega_\mathbf{k}} - \omega_\mathbf{k}(1+k_\perp^2) + 8k_x^2 \omega_\mathbf{k} \Big) \Big] \Big\}, \tag{43}$$

where $I_\mathbf{k} = 2\tilde{\varphi}_+ \tilde{\varphi}_- = 2|\tilde{\varphi}_+|^2$ is the intensity of pumping waves.

Substitution of Eqs. (42), (43) into (32) gives the following system of coupled linear equations for the mean electrostatic potential $\bar{\varphi}_0$ and parallel to external magnetic field motion $\bar{w}_0$:

$$\begin{cases} \bar{\varphi}_0 = I_\perp^\varphi \bar{\varphi}_0 + I_\perp^w \bar{w}_0, \\ \bar{w}_0 = I_{\rm II}^\varphi \bar{\varphi}_0 + I_{\rm II}^w \bar{w}_0. \end{cases} \tag{44}$$

Here

$$I_\perp^\varphi = -\frac{q_x^3}{1+q_x^2} \sum_\mathbf{k} \frac{k_y^2 I_\mathbf{k}}{D^{(1)2}} \Big\{ q_x \Big[\omega_\mathbf{k}^2 k_\perp^2 (1+k_\perp^2) - 4\omega_\mathbf{k}^2 k_x^2 k_\perp^2 - 8k_x^2 k_z^2 + k_z^2(1+2k_\perp^2) + \frac{k_z^4}{\omega_\mathbf{k}^2} \Big] - 2\Omega k_x \frac{k_z^2}{\omega_\mathbf{k}} \Big\}, \tag{45}$$

$$I_\perp^w = -\frac{q_x^3}{1+q_x^2} \sum_\mathbf{k} \frac{k_y^2 k_z I_\mathbf{k}}{D^{(1)2}} \Big\{ 2k_x \Omega (1+k_\perp^2) + q_x \Big[-\frac{k_z^2}{\omega_\mathbf{k}} - \omega_\mathbf{k}(1+k_\perp^2) + 8k_x^2 \omega_\mathbf{k} \Big] \Big\}, \tag{46}$$

$$I_{\rm II}^\varphi = -q_x^2 \sum_\mathbf{k} \frac{k_y^2 k_z I_\mathbf{k}}{\omega_\mathbf{k} D^{(1)2}} \Big\{ -\Omega^2 (1+k_\perp^2) + q_x^2 [k_z^2 + \omega_\mathbf{k}^2 (k_\perp^2 - 4k_x^2)] + 2\Omega q_x k_x \Big(\frac{k_z^2}{\omega_\mathbf{k}} - 2\omega_\mathbf{k} \Big) \Big\}, \tag{47}$$

$$I_{\rm II}^w = -q_x^2 \sum_\mathbf{k} \frac{k_y^2 I_\mathbf{k}}{D^{(1)2}} \Big\{ \Omega^2 (1+k_\perp^2)^2 + q_x^2 \omega_\mathbf{k} \Big(4k_x^2 \omega_\mathbf{k} - \frac{k_z^2}{\omega_\mathbf{k}} \Big) + 2\Omega q_x k_x \times \Big[\omega_\mathbf{k}(1+k_\perp^2) + \beta_n k_y \Big] \Big\}. \tag{48}$$

In these expressions, we can represent $D^{(1)}$ [see Eq. (41)] in the following way:

$$D^{(1)} = \big[2\omega_\mathbf{k}(1+k_\perp^2) - \beta_n k_y \big](\Omega - q_x V_g), \tag{49}$$

where

$$V_g = \frac{\partial \omega_\mathbf{k}}{\partial k_x} = -\frac{2k_x \omega_\mathbf{k}^2}{2\omega_\mathbf{k}(1+k_\perp^2) - \beta_n k_y}, \tag{50}$$

is the group velocity of pumping drift-ion-acoustic waves.

From the system (44), the following zonal flow dispersion equation follows

$$1 - (I_\perp^\varphi + I_{\rm II}^w) + I_\perp^\varphi I_{\rm II}^w - I_\perp^w I_{\rm II}^\varphi = 0. \tag{51}$$

Now, we deal with this equation for the monochromatic wave packet of the primary waves, i.e., we consider a single wave vector on the right-hand sides of Eqs. (45)–(48). Note that all these expressions are proportional to the second power of the small quantities (26). Thus, the right-hand sides of these expressions are relevant only in the case if the value $\Omega - q_x V_g$ is also small. Then, the coefficients (45)–(48) can be calculated at $\Omega \approx q_x V_g$. We find

$$(\Omega - q_x V_g)^2 I_\perp^\varphi|_{\Omega = q_x V_g} = -\frac{q_x^4 k_y^2 I_\mathbf{k}}{[\beta_n k_y - 2\omega_\mathbf{k}(1+k_\perp^2)]^3} \times \{ \beta_n^3 k_y^3 + \beta_n^2 k_y^2 \omega_\mathbf{k} [-5(1+k_\perp^2) + 8k_x^2 - k_\perp^2] + 4\beta_n k_y \omega_\mathbf{k}^2 \times [k_x^2(1-k_\perp^2) + 2(1+k_\perp^2)^2 + (1+k_\perp^2)(k_\perp^2 - 6k_x^2)] + 4\omega_\mathbf{k}^3 (1+k_\perp^2)[(1+k_\perp^2)(4k_x^2 - k_\perp^2) + k_x^2(2k_\perp^2 - 1) - (1+k_\perp^2)^2] \}, \tag{52}$$

$$(\Omega - q_x V_g)^2 I_\perp^w|_{\Omega = q_x V_g} = -\frac{q_x^4 k_y^2 k_z I_\mathbf{k}}{[\beta_n k_y - 2\omega_\mathbf{k}(1+k_\perp^2)]^3} \times \{ \beta_n^2 k_y^2 + 4\beta_n k_y \omega_\mathbf{k} [2k_x^2 - (1+k_\perp^2)] + 4\omega_\mathbf{k}^2 (1+k_\perp^2)[(1+k_\perp^2) - 3k_x^2] \}, \tag{53}$$

$$(\Omega - q_x V_g)^2 I_{II}^\varphi|_{\Omega = q_x V_g} = -\frac{q_x^4 k_y^2 k_z I_\mathbf{k}}{[\beta_n k_y - 2\omega_\mathbf{k}(1+k_\perp^2)]^4} \times \{ -\beta_n^3 k_y^3 - \beta_n^2 k_y^2 \omega_\mathbf{k} [1 + 8k_x^2 - 6(1+k_\perp^2)] + 4\beta_n k_y \omega_\mathbf{k}^2 \times [-2k_x^2 - 3(1+k_\perp^2)^2 + (1+k_\perp^2)(1+7k_x^2)] - 4\omega_\mathbf{k}^3 (1+k_\perp^2)[(1+k_\perp^2)(1+6k_x^2) - 3k_x^2 - 2(1+k_\perp^2)^2] \}, \tag{54}$$

$$(\Omega - q_x V_g)^2 I_{II}^w|_{\Omega = q_x V_g} = -\frac{q_x^4 k_y^2 \omega_\mathbf{k} I_\mathbf{k}}{[\beta_n k_y - 2\omega_\mathbf{k}(1+k_\perp^2)]^4} \times \{ \beta_n^3 k_y^3 + \beta_n^2 k_y^2 \omega_\mathbf{k} [8k_x^2 - 5(1+k_\perp^2)] + 4\beta_n k_y \omega_\mathbf{k}^2 (1+k_\perp^2)[-5k_x^2 + 2(1+k_\perp^2)] + 4\omega_\mathbf{k}^3 (1+k_\perp^2)^2 \times [3k_x^2 - (1+k_\perp^2)] \}. \tag{55}$$

With these expressions, we can show that last two terms in Eq. (51) cancel each other, so the zonal flow dispersion equation is

$$1-(I_\perp^\varphi+I_{\rm II}^w)=0. \tag{56}$$

Using Eqs. (52) and (55), we can find the following general expression for the squared zonal flow growth rate:

$$(\Omega-q_xV_g)^2=-\Gamma^2, \tag{57}$$

where

$$\begin{aligned}\Gamma^2=&\frac{q_x^4k_y^2\omega_{\mathbf{k}}^4I_{\mathbf{k}}}{[k_z^2+\omega_{\mathbf{k}}^2(1+k_\perp^2)]^4}\{[\beta_nk_y-2\omega_{\mathbf{k}}(1+k_\perp^2)]^4\\&+2\beta_n^3k_y^3\omega_{\mathbf{k}}(1+4k_x^2)\\&+4\omega_{\mathbf{k}}\beta_nk_yk_z^2[(1+k_\perp^2)(5+8k_x^2)+12k_x^2k_\perp^2]\\&+\omega_{\mathbf{k}}^2[\beta_n^2k_y^2-12\omega_{\mathbf{k}}^2(1+k_\perp^2)^2][(1+k_\perp^2)(9+4k_x^2)\\&+32k_x^2k_\perp^2]+12\omega_{\mathbf{k}}^4(1+k_\perp^2)^2\\&\times[(1+k_\perp^2)(8+3k_x^2)+29k_x^2k_\perp^2]\}.\end{aligned} \tag{58}$$

Let us consider the zonal flow generation by different branches of coupled drift and ion-acoustic pumping waves.

(1) In case of single drift waves branch (16), when $k_z=0$ and $\omega_1=\frac{\beta_nk_y}{1+k_\perp^2}$, we find from Eq. (58)

$$\Gamma^2=\frac{q_x^4k_y^2k_\perp^2I_{\mathbf{k}}}{(1+k_\perp^2)^2}\left(1+k_y^2-3k_x^2\right). \tag{59}$$

So the instability condition requires $1+k_y^2>3k_x^2$[16] and the fastest growth rate achieved when $k_x=0$.

(2) In the case of the slow branch of the coupled drift and ion-acoustic waves with the pumping frequency [see the second solution in Eq. (18)] $\omega_2=-\frac{k_z^2}{\beta_nk_y}(k_z\to0)$ yields

$$\Gamma^2=q_x^4k_y^2I_{\mathbf{k}}. \tag{60}$$

So, unlike (59), such instability needs no excitation condition (i.e., exists spontaneously) and the growth rate does not depend on k_z.

(3) In the case of single ion-acoustic branch (17), $\omega_{1,2}=\pm\frac{k_z}{(1+k_\perp^2)^{1/2}}$, $(\beta_n\to0)$ we get the following expression:

$$\Gamma^2=\frac{q_x^4k_y^2I_{\mathbf{k}}}{4(1+k_\perp^2)^2}\left[(1+k_\perp^2)\left(1+k_x^2+4k_y^2\right)-9k_x^2k_\perp^2\right]. \tag{61}$$

Thus, the generation does not depend on the sign of ω, i.e., it is the same for both ω_1 and ω_2. The instability condition requires

$$(1+k_\perp^2)(1+k_x^2+4k_y^2)>9k_x^2k_\perp^2, \tag{62}$$

and the fastest growth rate [as in the case (1)] is achieved when $k_x=0$. In addition as in the case (2), the growth rate is not influenced by k_z.

As to the parallel to the external magnetic field mean flow generation problem, we find from Eq. (44)

$$\frac{\bar w_0}{\bar\varphi_0}=\frac{I_{\rm II}^\varphi}{1-I_{\rm II}^w}. \tag{63}$$

Owing to the smallness of $I_{\rm II}^w\propto q_x^2$, we get approximately [using Eq. (57)]

$$\begin{aligned}\frac{\bar w_0}{\bar\varphi_0}=I_{II}^\varphi=&\frac{q_x^4k_y^2k_z\omega_{\mathbf{k}}^4I_{\mathbf{k}}}{\Gamma^2[k_z^2+\omega_{\mathbf{k}}^2(1+k_\perp^2)]^4}\\&\times\{-\beta_n^3k_y^3-\beta_n^2k_y^2\omega_{\mathbf{k}}[1+8k_x^2-6(1+k_\perp^2)]\\&+4\beta_nk_y\omega_{\mathbf{k}}^2\left[(1+k_\perp^2)(1+7k_x^2)-3(1+k_\perp^2)^2-2k_x^2\right]\\&-4\omega_{\mathbf{k}}^3(1+k_\perp^2)[(1+k_\perp^2)(1+6k_x^2)-2(1+k_\perp^2)^2-3k_x^2]\}.\end{aligned} \tag{64}$$

(1) In case of the slow branch, $\omega_2=-\frac{k_z^2}{\beta_nk_y}$, $(k_z\to0)$ Eq. (64) yields

$$\frac{\bar w_0}{\bar\varphi_0}=-\frac{k_z}{\beta_nk_y}. \tag{65}$$

(2) In case of the pure ion-acoustic waves $\omega_{1,2}=\pm\frac{k_z}{(1+k_\perp^2)^{1/2}}$ $(\beta_n=0)$, we get the excitation

$$\frac{\bar w_0}{\bar\varphi_0}=\pm\frac{1}{\sqrt{1+k_\perp^2}}\frac{1+3k_y^2+2k_y^4-4k_x^4-2k_x^2k_y^2}{1+2k_x^2+5k_y^2+4k_y^4-8k_x^4-4k_x^2k_y^2}. \tag{66}$$

Note that in this case, the ratio (65) does not depend on k_z.

B. Large-scale structures, $k_\perp\rho_s\ll1$

Now we are going to investigate the zonal flow excitation problem by large scale pumping drift- ion-acoustic waves which is described by the system (24). Note that this system contains two scalar and one vector nonlinearities. Of course, the linear Eqs. (27)–(31) remain the same. The same will also remain Eq. (32) for $R_{\rm II}$ and $r_{\rm II}(\mathbf{k})$, where new solutions for sideband amplitudes should be insert. As to $R_\perp$ now we have the following modified Eq. (32):

$$\begin{aligned}-i\Omega\bar\varphi_0=R_\perp=&\frac{1}{1+q_x^2}\sum_{\mathbf{k}}r_\perp(\mathbf{k})\\=&\frac{1}{1+q_x^2}\sum_{\mathbf{k}}(\tilde\varphi_-\hat\chi_+-\tilde\varphi_+\hat\chi_-),\end{aligned} \tag{67}$$

where

$$\hat\chi_\pm=\pm q_x^2(2k_xk_y\pm q_xk_y)\hat\varphi_\pm-ik_z\left(\frac{k_z}{\omega_{\mathbf{k}}}\hat\varphi_\pm-\hat w_\pm\right). \tag{68}$$

We can find that now sideband amplitudes satisfy the following system:

$$\begin{cases} \left[\omega_\pm\left(1+k_{\perp\pm}^2\right)\mp\beta_n k_y\right]\hat{\varphi}_\pm \mp k_z \hat{w}_\pm = \mp i\left(q_x^2-k_\perp^2\right)k_y q_x \tilde{\varphi}_\pm \bar{\varphi}_0 \mp \beta_T k_y \tilde{\varphi}_\pm \bar{\varphi}_0 \pm k_z \bar{w}_0 \tilde{\varphi}_\pm, \\ \pm k_z \hat{\varphi}_\pm - \omega_\pm \hat{w}_\pm = \pm i\left(\bar{w}_0 - \dfrac{k_z}{\omega_{\mathbf{k}}}\bar{\varphi}_0\right)k_y q_x \tilde{\varphi}_\pm. \end{cases} \tag{69}$$

Solutions of the system (69) are

$$\begin{cases} \hat{\varphi}_\pm = \dfrac{\tilde{\varphi}_\pm}{D_\pm}\left\{k_z(-ik_y q_x \pm \omega_\pm)\bar{w}_0 + \bar{\varphi}_0 k_y\left[i\dfrac{k_z^2}{\omega_{\mathbf{k}}}q_x \mp \omega_\pm\left[i\left(q_x^2-k_\perp^2\right)q_x+\beta_T\right]\right]\right\}, \\ \hat{w}_\pm = \dfrac{\tilde{\varphi}_\pm}{D_\pm}\left\{\bar{w}_0\left[k_z^2 \mp i k_y q_x\left(\omega_\pm\left(1+k_{\perp\pm}^2\right)\mp\beta_n k_y\right)\right] + \bar{\varphi}_0\left[\pm i\dfrac{k_z}{\omega_{\mathbf{k}}}k_y q_x\left[\omega_\pm\left(1+k_{\perp\pm}^2\right)\mp\beta_n k_y\right] - k_z k_y\left[i\left(q_x^2-k_\perp^2\right)q_x+\beta_T\right]\right]\right\}, \end{cases} \tag{70}$$

where $D_\pm$ is defined by Eq. (37), so Eqs. (40) and (41) also remain valid.

As previously, using the expansion over the small parameters (26) and keeping only necessary main terms, we find the following zonal flow driving forces:

$$\begin{aligned} R_\perp = \frac{q_x^2}{1+q_x^2}\sum_{\mathbf{k}}\frac{k_y I_{\mathbf{k}}}{D^{(1)2}}\Bigg\{ & \bar{\varphi}_0 k_y\Bigg[iq_x^2\left(\frac{k_z^2}{\omega_{\mathbf{k}}}+\omega_{\mathbf{k}}k_\perp^2\right) \\ & \times\left(2q_x k_x\omega_{\mathbf{k}}^2+2\Omega\omega_{\mathbf{k}}\left(1+k_\perp^2\right)-\Omega\beta_n k_y\right) \\ & -\Omega\beta_T\left(2\Omega k_x\frac{k_z^2}{\omega_{\mathbf{k}}}+q_x\omega_{\mathbf{k}}^2\left(1+k_\perp^2-4k_x^2\right)+q_x k_z^2\right)\Bigg] \\ & +\bar{w}_0 k_z\Bigg[\Omega\left(2\Omega k_x\frac{k_z^2}{\omega_{\mathbf{k}}}+q_x\omega_{\mathbf{k}}^2\left(1+k_\perp^2\right)\right. \\ & \left.+q_x k_z^2-4\omega_{\mathbf{k}}^2 q_x k_x^2\right) \\ & -iq_x^2 k_y\left(2q_x k_x\omega_{\mathbf{k}}^2+2\Omega\omega_{\mathbf{k}}\left(1+k_\perp^2\right)-\Omega\beta_n k_y\right)\Bigg]\Bigg\}, \end{aligned} \tag{71}$$

$$\begin{aligned} R_{\mathrm{II}} = \Omega q_x\sum_{\mathbf{k}}\frac{k_y k_z I_{\mathbf{k}}}{\omega_{\mathbf{k}}D^{(1)2}}\{ & -\beta_T k_y\bar{\varphi}_0\left[\Omega^2\left(1+k_\perp^2\right)\right. \\ & \left.+q_x^2\omega_{\mathbf{k}}^2+4\Omega\omega_{\mathbf{k}}q_x k_x\right]+k_z\bar{w}_0\left[\Omega^2\left(1+k_\perp^2\right)\right. \\ & \left.+q_x^2\omega_{\mathbf{k}}^2+4\Omega\omega_{\mathbf{k}}q_x k_x\right]\}. \end{aligned} \tag{72}$$

Now we use relations (32) to obtain the system (44), where

$$I_\perp^\varphi = -iq_x^2\sum_{\mathbf{k}}\frac{k_y^2 I_{\mathbf{k}}}{D^{(1)2}}\beta_T\left[2\Omega k_x\frac{k_z^2}{\omega_{\mathbf{k}}}+q_x\omega_{\mathbf{k}}^2\left(1+k_\perp^2-4k_x^2\right)+q_x k_z^2\right], \tag{73}$$

$$I_\perp^w = iq_x^2\sum_{\mathbf{k}}\frac{k_y k_z I_{\mathbf{k}}}{D^{(1)2}}\left[2\Omega k_x\frac{k_z^2}{\omega_{\mathbf{k}}}+q_x\omega_{\mathbf{k}}^2\left(1+k_\perp^2-4k_x^2\right)+q_x k_z^2\right], \tag{74}$$

$$I_{\mathrm{II}}^\varphi = -iq_x\sum_{\mathbf{k}}\frac{k_y^2 k_z\beta_T I_{\mathbf{k}}}{\omega_{\mathbf{k}}D^{(1)2}}\left[\Omega^2\left(1+k_\perp^2\right)+q_x^2\omega_{\mathbf{k}}^2+4\Omega\omega_{\mathbf{k}}q_x k_x\right], \tag{75}$$

$$I_{\mathrm{II}}^w = iq_x\sum_{\mathbf{k}}\frac{k_y k_z^2 I_{\mathbf{k}}}{\omega_{\mathbf{k}}D^{(1)2}}\left[\Omega^2\left(1+k_\perp^2\right)+q_x^2\omega_{\mathbf{k}}^2+4\Omega\omega_{\mathbf{k}}q_x k_x\right]. \tag{76}$$

As it was explained previously we need the coefficients (73)–(76) at the value $\Omega = q_x V_g$, where the group velocity V_g is defined by Eq. (50). So we have

$$\begin{aligned} & \left(\Omega - q_x V_g\right)^2 I_\perp^\varphi|_{\Omega=q_x V_g} \\ & = -i\frac{q_x^3 k_y^2 I_{\mathbf{k}}\beta_T}{\left[\beta_n k_y - 2\omega_{\mathbf{k}}\left(1+k_\perp^2\right)\right]^3}\{4k_x^2 k_z^2\omega_{\mathbf{k}}+\beta_n k_y k_z^2 \\ & \quad -\omega_{\mathbf{k}}^3\left(1+k_\perp^2\right)\left(1+k_\perp^2-4k_x^2\right) \\ & \quad -2\omega_{\mathbf{k}}\left(1+k_\perp^2\right)k_z^2-\omega_{\mathbf{k}}\left(1+k_\perp^2-4k_x^2\right)k_z^2\}, \end{aligned} \tag{77}$$

$$\begin{aligned} & \left(\Omega - q_x V_g\right)^2 I_\perp^w|_{\Omega=q_x V_g} \\ & = i\frac{q_x^3 k_y k_z I_{\mathbf{k}}}{\left[\beta_n k_y - 2\omega_{\mathbf{k}}\left(1+k_\perp^2\right)\right]^3}\{4k_x^2 k_z^2\omega_{\mathbf{k}}+\beta_n k_y k_z^2 \\ & \quad -\omega_{\mathbf{k}}^3\left(1+k_\perp^2\right)\left(1+k_\perp^2-4k_x^2\right) \\ & \quad -2\omega_{\mathbf{k}}\left(1+k_\perp^2\right)k_z^2-\omega_{\mathbf{k}}\left(1+k_\perp^2-4k_x^2\right)k_z^2\}, \end{aligned} \tag{78}$$

$$\begin{aligned} & \left(\Omega - q_x V_g\right)^2 I_{\mathrm{II}}^\varphi|_{\Omega=q_x V_g} \\ & = -i\frac{q_x^3 k_y^2 k_z\omega_{\mathbf{k}}\beta_T I_{\mathbf{k}}}{\left[\beta_n k_y - 2\omega_{\mathbf{k}}\left(1+k_\perp^2\right)\right]^4}\{\beta_n^2 k_y^2-4\omega_{\mathbf{k}}\beta_n k_y \\ & \quad \times\left(1+k_\perp^2-2k_x^2\right) \\ & \quad +4\omega_{\mathbf{k}}^2\left(1+k_\perp^2\right)\left(1+k_\perp^2-3k_x^2\right)\}, \end{aligned} \tag{79}$$

$$\begin{aligned} \left(\Omega - q_x V_g\right)^2 I_{II}^w|_{\Omega=q_x V_g} = i & \frac{q_x^3 k_z^2 k_y\omega_{\mathbf{k}}I_{\mathbf{k}}}{\left[\beta_n k_y - 2\omega_{\mathbf{k}}(1+k_\perp^2)\right]^4} \\ & \times\{\beta_n^2 k_y^2-4\omega_{\mathbf{k}}\beta_n k_y(1+k_\perp^2-2k_x^2) \\ & +4\omega_{\mathbf{k}}^2(1+k_\perp^2)(1+k_\perp^2-3k_x^2)\}. \end{aligned} \tag{80}$$

Now we can show that the zonal flow dispersion equation (56) remains valid. Thus, we can get the following expression for the squared zonal flow growth rate:

$$(\Omega - q_x V_g)^2 = i\frac{q_x^3 k_y I_{\mathbf{k}} \omega_{\mathbf{k}}^4}{\left[k_z^2 + \omega_{\mathbf{k}}^2(1+k_\perp^2)\right]^4} \times \{\beta_T(1+k_\perp^2)\omega_{\mathbf{k}} k_y k_z^2(20k_x^2\omega_{\mathbf{k}} + 5\beta_n k_y) - \omega_{\mathbf{k}}^2\beta_T k_y(1+k_\perp^2)^2[7k_z^2 + \omega_{\mathbf{k}}^2(1+k_\perp^2) - 4k_x^2\omega_{\mathbf{k}}^2)] - \beta_T k_z^2\beta_n k_y^2(\beta_n k_y + 8k_x^2\omega_{\mathbf{k}}) + \beta_n^2 k_y^2 k_z^2\omega_{\mathbf{k}} - 4\omega_{\mathbf{k}}^2 k_z^2\beta_n k_y(1+k_\perp^2 - 2k_x^2) + 4\omega_{\mathbf{k}}^3 k_z^2(1+k_\perp^2)(1+k_\perp^2 - 3k_x^2)\}. \tag{81}$$

Let us consider the following cases:

(1) In case of single drift waves, when $k_z = 0$ and $\omega_1 = \frac{\beta_n k_y}{1+k_\perp^2}$, we find from Eq. (81)

$$(\Omega - q_x V_g)^2 = -i q_x^3 k_y^2 \beta_T I_{\mathbf{k}}. \tag{82}$$

(2) Other slow branch $\omega_2 = -\frac{k_z^2}{\beta_n k_y}$, $(k_z \to 0)$ yields

$$(\Omega - q_x V_g)^2 = -i\frac{q_x^3 \beta_T k_z^2}{\beta_n^2} I_{\mathbf{k}}. \tag{83}$$

(3) In the case of single ion-acoustic branch $\omega_{1,2} = \pm\frac{k_z}{(1+k_\perp^2)^{1/2}}$, $(\beta_n \to 0)$, we get the following expression from (81):

$$(\Omega - q_x V_g)^2 = -i\frac{1}{4} q_x^3 k_y I_{\mathbf{k}}(2\beta_T k_y \mp k_z). \tag{84}$$

Parallel to the magnetic field mean flow generation rate, we can find analogously to Eq. (64)

$$\frac{\bar{w}_0}{\bar{\varphi}_0} = I_{\mathrm{II}}^{\varphi} = -i\frac{q_x^3 k_y^2 k_z \omega_{\mathbf{k}}^5 \beta_T I_{\mathbf{k}}}{(\Omega - q_x V_g)^2\left[k_z^2 + \omega_{\mathbf{k}}^2(1+k_\perp^2)\right]^4} \times \{\beta_n^2 k_y^2 - 4\omega_{\mathbf{k}}\beta_n k_y(1+k_\perp^2 - 2k_x^2) + 4\omega_{\mathbf{k}}^2(1+k_\perp^2)(1+k_\perp^2 - 3k_x^2)\}. \tag{85}$$

Here, the value of $\Omega - q_x V_g$ is defined by Eq. (81). We see that mean flow $\bar{w}_0$ is exciting only when $k_z \neq 0$.

(1) For the case (83), we get

$$\frac{\bar{w}_0}{\bar{\varphi}_0} = -\frac{k_z}{\beta_n k_y}. \tag{86}$$

We see that the ratio is not influenced by the temperature gradient β_T.

(2) For the case of ion-acoustic branch (84), we get

$$\frac{\bar{w}_0}{\bar{\varphi}_0} = \pm\frac{k_y \beta_T}{2\beta_T k_y \mp k_z}. \tag{87}$$

So again, ω_1 and ω_2 generation has the opposite sign.

VI. CONCLUSION

The analysis given in the presented paper shows how sheared zonal flows are generated by low-frequency coupled electrostatic drift and ion-acoustic waves. According to laboratory plasma experiments,[5,21,22] main attention to large-scale ($k_\perp \rho_s \ll 1$) drift-ion-acoustic waves is given. Carried out investigation provides an essential nonlinear mechanism for the spectral energy transfer from small-scale drift-ion-acoustic waves to large-scale enhanced zonal flows.

In Sec. II valid for arbitrary wavelengths of primary waves basic system of nonlinear equations consisting of the general HM equation for electrostatic potential and equation describing parallel to magnetic field ions motion is obtained [see Eqs. (9) and (10)].

In Sec. III, a linear regime of coupled drift-ion-acoustic waves is given in detail and different limiting frequencies are obtained [see Eqs. (16)–(19)]. In consequence of coupling with drift waves, ion-acoustic waves become dispersive [see Eq. (17)].

In Sec. IV, primary waves of different wavelength-scales are considered and the appropriate system of nonlinear equations is obtained. Namely, it is shown that for the small and intermediate wavelengths ($k_\perp \rho_s \geq 1$), the dynamical system of Eq. (22) is valid which contains only vector nonlinearity and respectively describes dipole structures. In the case of large-scale wavelengths ($k_\perp \rho_s \ll 1$), electrons temperature gradient effects become important forming KdV type scalar nonlinearity along with other nonlinearities coming from the ions parallel to magnetic field motion [see Eq. (24)].

In Sec. V, the generation of sheared zonal flows by comparatively small-scale electrostatic coupled drift-ion-acoustic waves in laboratory plasmas is investigated. The generation is due to the parametric excitation under the three-wave interaction, in which the coupling between the pump electrostatic drift-ion-acoustic waves and side-band modes generates large-scale modes, so called zonal flows [see Eq. (25)]. Actually the zonal flow is spontaneously generated from the pumping drift-ion-acoustic wave fluctuations via the action of electrostatic stress $R_\perp$ and electromotive force R_{II} [see Eq. (32)]. To describe the process system of basic equations (22) and (24) is used. Corresponding general expressions for the squared zonal flow growth rate are obtained in the case of small and intermediate ($k_\perp \rho_s \geq 1$) and large-scale ($k_\perp \rho_s \ll 1$) pumping waves [see Eqs. (57), (58), and (81)]. It is shown that in the case of small and intermediate excitation vector nonlinearity plays the main role and for the generation necessary excitation conditions are needed. It is found that the wave vector of the fastest growing mode is perpendicular to that of the drift-ion-acoustic pump wave [see Eqs. (59) and (61)]. In the case of large-scale excitation due to the electrons temperature gradient scalar nonlinearities became responsible for the sheared zonal flow generation and in contrast to small-scale turbulence no generation conditions are needed. In addition the growth rate in the case of large-scale excitation is more ($\sim q_x^{3/2}$) compared to the small and intermediate excitation ($\sim q_x^2$). That is why observed in laboratory experiments large-scale drift waves fluctuations spontaneously excite the sheared zonal flow. Explicit expressions for the maximum growth rate are obtained in particular, cases [see Eqs. (59)–(61) and (82)–(84)].

In addition, we have shown that owing to the existence of ion-acoustic waves mean sheared flow of ions can be generated along the equilibrium toroidal magnetic field. General expressions for such flows are obtained both for the intermediate and large-scale coupled drift-ion-acoustic waves [see Eqs. (64) and (85)]. Corresponding explicit expressions for such toroidal flow are obtained in particular cases [see Eqs. (65), (66), (86), and (87)].

Let us estimate obtained growth rates numerically.

In the case of small and intermediate-scale structures ($k_\perp \rho_s \geq 1$), we get the following estimation for the zonal flow growth rate from obtained Eqs. (59)–(61):

$$\gamma \sim \omega_{ci}\left(\frac{q_x}{k_y}\right)^2 (k_y\rho_s)^3 \left|\frac{e\tilde{\varphi}_+}{T_e}\right|. \tag{88}$$

For the chosen experimental observations[5,21,22]

$$\omega_{ci} \sim 10^8 \mathrm{s}^{-1}, \quad k_y\rho_s \sim 5, \quad |e\tilde{\varphi}_+/T_e| \sim 10^{-1}, \quad q_x/k_y \sim 10^{-1},$$

we get $\gamma \approx 10^7\ \mathrm{s}^{-1}$, which is consistent with existing observations. As to the amplitude of toroidal zonal flow, from Eq. (66), we have the following estimation:

$$\frac{\bar{w}_0}{\bar{\varphi}_0} \sim \frac{1}{k_y\rho_s}. \tag{89}$$

So such as toroidal zonal flow is only several times less than poloidal one.

From obtained Eqs. (82)–(84), in the case of large-scale structures ($k_\perp \rho_s \ll 1$), we note the importance of electrons temperature space inhomogeneity for the unconditional zonal flow generation. We get the following estimation for the zonal flow growth rate:

$$\gamma \sim \omega_{ci}\left(\frac{q_x}{k_y}\right)^{3/2} (k_y\rho_s)^{5/2} (\beta_T\rho_s)^{1/2} \left|\frac{e\tilde{\varphi}_+}{T_e}\right|. \tag{90}$$

For the typical experimental data[5,21,22]

$$\omega_{ci} \sim 10^8 \mathrm{s}^{-1}, \quad k_y\rho_s \sim 0.3, \quad \beta_T\rho_s \sim \frac{\rho_s}{L_T} \sim \frac{0.1}{10} = 10^{-2},$$
$$\frac{e\tilde{\varphi}_+}{T_e} \sim 10^{-1}, \quad \frac{q_x}{k_y} \sim 0.1,$$

we get $\gamma \sim 10^3\ \mathrm{s}^{-1}$, which is 10^4 times smaller than that obtained in the small-scale turbulence [see Eq. (88)]. However, it seems that the large-scale case of turbulence is more typical for laboratory plasmas.[5,21,22] As to the amplitude of toroidal zonal flow in the case of large-scale turbulence ($k_\perp \rho_s \ll 1$) from Eq. (87), we get the estimation

$$\frac{\bar{w}_0}{\bar{\varphi}_0} \sim 1. \tag{91}$$

Thus, the excited toroidal zonal flow amplitude is of order of the poloidal one. Observe that typical drift waves frequency value $\omega \sim k_y v^* \sim 10^5$–$10^7 \mathrm{s}^{-1}$.

[1]L. I. Rudakov and R. Z. Sagdeev, Sov. Phys. Dokl. **6**, 415 (1961).
[2]A. B. Mikhailovskii, *Theory of Plasma Instabilities: Volume 2: Instabilities of an Inhomogeneous Plasma* (Springer US, 2014).
[3]A. B. Mikhailovskii, *Electromagnetic Instabilities in an Inhomogeneous Plasma* (Taylor & Francis, 1992).
[4]A. B. Mikhailovskii, "Instabilities in inhomogeneous plasma," in *Basic Plasma Physics Handbook*, edited by A. A. Galeev and R. N. Sudan (North Holland Publisher, Amsterdam, 1983), Vol. 1, pp. 587–610.
[5]W. Horton, "Drift waves turbulence and anomalous transport," in *Basic Plasma Physics Handbook*, edited by A. A. Galeev and R. N. Sudan (North Holland Publisher, New York, 1984), Vol.2, pp. 383–449.
[6]A. Hasegawa and K. Mima, Phys. Rev. Lett. **39**, 205 (1977).
[7]A. Hasegawa and K. Mima, Phys. Fluids **21**, 87 (1978).
[8]V. I. Petviashvili, Fiz. Plasmy **3**, 270 (1977); V. I. Petviashvili, Sov. J. Plasma Phys. **3**, 150 (1977).
[9]V. I. Petviashvili, Pis'ma ZHETF **32**, 632 (1980); V. I. Petviashvili, JETP Lett. **32**, 619 (1980).
[10]A. B. Mikhailovskii, "Vortices in plasma and hydrodynamics," in *Nonlinear Phenomena in Plasma Physics and Hydrodynamics*, edited by R. Z. Sagdeev (Mir Publishers, Moscow, 1986).
[11]M. V. Nezlin and G. P. Chernikov, Plasma Phys. Rep. **21**, 922 (1995).
[12]L. Chen, Z. Lin, and R. White, Phys. Plasmas **7**, 3129 (2000).
[13]P. N. Guzdar, R. G. Kleva, and L. Chen, Phys. Plasmas **8**, 459 (2001).
[14]P. K. Shukla and L. Stenflo, Eur. Phys. J. D **20**, 103 (2002).
[15]C. N. Lashmore-Davies, A. Thyagaraja, and D. R. McCarthy, Phys. Plasmas **12**, 122304 (2005).
[16]T. D. Kaladze, D. J. Wu, O. A. Pokhotelov, R. Z. Sagdeev, L. Stenflo, and P. K. Shukla, Phys. Plasmas **12**, 122311 (2005).
[17]T. D. Kaladze, M. Shad, and L. V. Tsamalashvili, Phys. Plasmas **17**, 022304 (2010).
[18]A. I. Smolyakov, P. H. Diamond, and V. I. Shevchenko, Phys. Plasmas **7**, 1349 (2000).
[19]A. I. Smolyakov, P. H. Diamond, and M. Malkov, Phys. Rev. Lett. **84**, 491 (2000).
[20]A. B. Mikhailovskii, A. I. Smolyakov, E. A. Kovalishen, M. S. Shirokov, V. S. Tsypin, P. V. Botov, and R. M. O. Galvão, Phys. Plasmas **13**, 042507 (2006).
[21]W. Horton, Rev. Mod. Phys. **71**, 735 (1999).
[22]W. Horton and A. Hasegawa, Chaos **4**, 227 (1994).
[23]J. D. Meiss and W. Horton, Phys. Fluids **26**, 990 (1983).
[24]S. V. Antipov, M. V. Snezhkin, and A. S. Trubnikov, Sov. Phys. - JETP **55**, 85 (1982).
[25]T. Kaladze, J. Rogava, L. Tsamalashvili, and M. Tsiklauri, Phys. Lett. A **343**, 199 (2005).
[26]T. D. Kaladze, M. Shad, and H. A. Shah, Phys. Plasmas **16**, 024502 (2009).
[27]T. D. Kaladze, O. A. Pokhotelov, and M. Shad, J. Plasma Phys. **76**, 635 (2010).
[28]T. D. Kaladze and O. Kharshiladze, Phys. Plasmas **23**, 122306 (2016).
[29]T. D. Kaladze, D. J. Wu, and L. Yang, Phys. Plasmas **14**, 032305 (2007).